ELEMENTARY ALGEBRA

SLAUGHT AND LENNES

80
90
94
80
80
100
100
100
100
a
a
a+
a+
100
a+
a+(last)

odd no. of negative signs = −
even no. = +

Row 2. No. 3

99

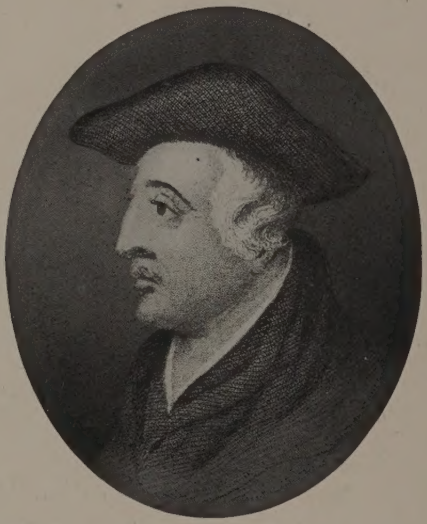
X ÷ + − in their order

Algebra test

Miss Elizabeth E. Evans
"Freshman"
B.C.H.S.
1924–25.

Betty Evaline Evans
Beach City
High School.

Ed Bixler
Bolivar W.S.



Roger Bacon

(1214–1294)

"All sciences rest ultimately on mathematics."

"Mathematics should be regarded as the alphabet of all philosophy."

"Divine mathematics, which alone can purge the intellect and fit the student for the acquirement of all knowledge."

ELEMENTARY ALGEBRA

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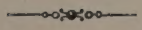

ALLYN AND BACON

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EAT

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PREFACE

The *Elementary Algebra* is planned to cover the work of the first year in this subject. It is in no sense a revision of the authors' First Principles of Algebra, but a new book, designed to meet the most exacting requirements of college entrance or other examination boards and the syllabi of various states. The presentation of topics, therefore, follows the traditional order.

The new Algebra contains numerous attractive features, all aiming to make the subject more simple and interesting and therefore more valuable to first year pupils. Among these features the most distinctive are perhaps the following four:

The presentation of the subject is as simple as it can be made.

The book is equipped with an unusually full and complete set of exercises and problems.

Vital purpose is given to the study of algebra by using it to do interesting and useful things.

Emphasis is given to the human interest of algebra.

To take up these four features in detail:

The simplicity of the book is especially shown (*a*) in the careful explanation of each new subject, which is fully discussed, defined, and illustrated; (*b*) in the gradation of topics, — so carefully managed as to be almost imperceptible to the pupil; and (*c*) in the gradual and natural introduction of the symbols and processes of algebra.

Each new topic is fixed by a large number of simple illustrative examples and exercises, oral and written, and finally

by practical problems. These exercises are unusually numerous and thorough, and there is at the end of the book a complete set of reviews for every chapter, followed by twenty pages of miscellaneous problems.

The interest and value of the subject are emphasized by showing (*a*) how algebra grows out of arithmetic, (*b*) how much easier it is to solve problems by algebra than by arithmetic, and (*c*) by applying these algebraic solutions to countless problems of every-day experience, as well as to those taken from other studies in the school.

To emphasize the human interest of algebra, numerous historical notes have been introduced, showing how many scholars have labored through the centuries to develop this subject. Also there are portraits of well-known mathematicians, with accounts of their lives and of their contributions to mathematics, especially to algebra. A briefer course may be made, if desired, by the omission of certain topics which are marked in the text with a star. These omissions will not interrupt the continuity of the work.

It is hoped that the *Elementary Algebra* will be found a practical and satisfactory instrument for the study of the subject, and that it will still further extend the *concrete study of algebra*, with which the authors have been identified.

H. E. S.
N. J. L.

JUNE, 1915.

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ELEMENTARY ALGEBRA

CHAPTER I

INTRODUCTION TO ALGEBRA

LETTERS USED TO REPRESENT NUMBERS

1. Algebra compared with Arithmetic. — Algebra, like arithmetic, deals with numbers. In arithmetic numbers are represented by means of the Arabic **numerals** 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. In algebra **letters**, as well as numerals, are used to represent numbers.

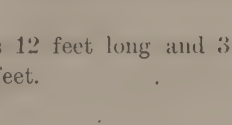
To illustrate this: If a blackboard is 12 feet long and 3 feet wide, its area is $12 \times 3 = 36$ square feet.

In *arithmetic* we write the rule:

$$area = length \times width.$$

In *algebra* this rule is abbreviated to read:

$$a = l \times w,$$



where l and w represent the **number of units** in the length and width respectively, and a the **number of square units** in the area.

But in algebra the sign $\times$ is usually omitted between two letters, so that this rule is written:

$$a = lw.$$

Formula. The expression, $a = lw$, is called a *formula*.

A formula is a rule expressed in algebraic symbols.

Thus, $a = lw$ is a formula giving the area of a rectangle.

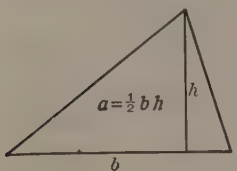
So we see that one advantage in using letters to represent numbers is to shorten the rules of arithmetic.

2. **The systematic use of letters to represent numbers** is one of the chief points of difference between algebra and arithmetic. In the following oral and written exercises note the simplicity with which the rules of arithmetic may be stated as formulas by means of letters.

ORAL EXERCISES

1. How many square inches in a slate 8 inches wide and 10 inches long? In one 6 inches wide and 9 inches long? In one w inches wide and l inches long?

2. How many square feet in the top of a table 3 feet wide and 5 feet long? In one 2 feet wide and 3 feet long? In one a feet wide and b feet long?



In arithmetic we learned that the area of a triangle is one half the product of the base and the altitude, or height. Thus, the area of a triangle is represented by the formula:

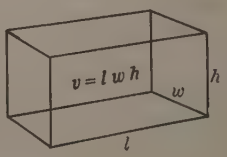
$$a = \tfrac{1}{2}bh.$$

3. How many square inches in a triangle whose base is 10 inches and whose altitude is 4 inches? In one whose base is 8 inches and altitude 6 inches?

In arithmetic we learned that the volume of a rectangular solid is the product of the length, width, and height; that is,

$$v = lwh.$$

4 How many cubic inches in a box 3 inches wide, 5 inches long, and 2 inches high? In one 4 inches wide, 5 inches long, and 3 inches high? In one w inches wide, l inches long, and h inches high?



5. How many cubic yards in an excavation 15 yards long, 10 yards wide, and 2 yards deep?

3. Symbols of Operation. The signs of operation, $+$, $-$, $\times$, $\div$, and the sign of equality, $=$, are used in algebra with the same meaning that they have in arithmetic. However, multiplication is often indicated by a point written above the line or by omitting the sign altogether, as in the preceding formulas.

Thus, $2 \times l \times w$ may be written $2 \cdot l \cdot w$ or $2\,lw$.
Here letters differ from numerals; for 25 means $20 + 5$, not $2 \cdot 5$.

WRITTEN EXERCISES

1. Write a formula giving the area of a rectangle. State this formula in words.

2. Using the formula $a = lw$, find the area in square rods of a field 35 rods long and 25 rods wide.

3. Using the same formula, find the area of a rectangle 17 feet wide and 20 feet long.

4. Using the formula $a = \frac{1}{2}\,bh$, find the area of a triangle whose base is 16 feet and whose altitude is 11 feet. State the process in words.

5. Using the same formula, find the area of a triangle whose base is 37 rods and whose altitude is 23 rods.

6. Given the length, width, and height of a rectangular solid, how do you find its volume? State this rule as a formula, using l, w, h, and v to represent the length, width, height, and volume, respectively. State it also in words.

7. Using the formula $v = lwh$, find the volume of a coal bin 15 feet long, 12 feet wide, and 9 feet high.

8. How many cubic feet will a freight car hold which is 40 feet long, 8 feet wide, and 7 feet high?

9. An excavation for a basement is 45 feet long, 28 feet wide, and 8 feet deep. How many cubic feet in it?

WRITTEN EXERCISES

By arithmetic 8 per cent of 90 means $.08 \times 90 = 7.2$. In this example 90 is called the **base**, .08 the **rate**, and 7.2 the **percentage**.

1. In like manner find 6 per cent of 125; 5 per cent of 350; 3 per cent of 80; and 10 per cent of 4.9. In each case how is the percentage found?

The rule for finding the percentage is stated in the following formula, in which b represents the base, r the rate, and p the percentage:

$$p = br.$$

2. Using the formula $p = br$, find the percentage if $b = 64.8$, $r = 16\%$; also if $b = 1080$, $r = 7\%$.

By a rule of arithmetic the interest on \$ 500 for 3 years at 6 per cent is $500 \times .06 \times 3 = 90$. In this example 500 is the **principal**, .06 the **rate**, and 3 the **time**.

3. In like manner, find the interest on \$ 900 for 7 years at 4 per cent, on \$ 1250 for 5 years at 5 per cent, on \$ 800 for 10 years at 6 per cent.

The rule for finding the interest is stated in the following formula, in which p represents the principal, r the rate, t the time, and i the interest:

$$i = prt.$$

4. Using the formula, $i = prt$, find the interest on \$ 1500 for 12 years at 6 per cent. Also find i if $p = 1750$, $r = 5\%$, and $t = 4$.

5. Find the area of a rectangle 24 feet long and 18 feet wide. State the formula used.

6. Find the volume of a rectangular coal bin 14 feet long, 10 feet wide, and 6 feet deep. State the formula used.

7. Find the percentage if the base is 74 and the rate 14 per cent. State the formula used.

8. Find the interest if the principal is \$ 840, the rate 6 per cent, and the time 3 years. State the formula used.

HISTORICAL NOTE

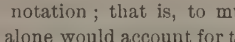
The Arabic or Hindu Numerals. The Arabs brought our present system of numerals to Europe when they invaded Spain from Morocco in the eighth century of our era. It is now known that these numerals, including zero, are really of Hindu origin and that they were invented between the years 200 and 650 A.D.

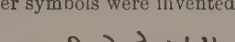
In the eleventh and twelfth centuries the Mohammedan Arabs, who had settled in Spain and Northern Africa, traded with the merchants from the great Italian cities, and in this way their system of numerals gradually spread over Europe.

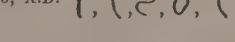
In 1202 Leonardo Fibonacci of Pisa wrote a book on mathematics, the *Liber Abaci*, in which he explained the Arabic numerals for the first time to the people of Italy. He says that he published this book "in order that the Latin (Italian) race might no longer be deficient in that knowledge." The use of the new system spread but slowly, however, and it was not until the seventeenth century that it came to be used exclusively in Germany and England.

The Greeks and Romans had used letters for number symbols, but the great step in advance was taken by the Hindus when they introduced the principle of *place value* which was made possible by the symbol for zero. Thus, in $51 = 50 + 1$, the 5 means 5 tens, while in $VI = V + 1$, the V means simply five ones.

Of all mathematical inventions none has contributed more to general intelligence and material welfare than this one. To appreciate this, one needs only to try to multiply two such numbers as 589 and 642 when expressed in the Roman notation; that is, to multiply DLXXXIX by DCXLII. This difficulty alone would account for the very slow development of algebra until better symbols were invented.

Indian Numerals about 950, A.D. 

Arabic Numerals about 1100, A.D. 

Of German origin about 1385, A.D. 

Probably Italian about 1400, A.D.

ALGEBRAIC EXPRESSIONS

4. Algebraic Expressions. We have seen that in algebra letters as well as numerals are used to represent numbers. We have seen also that the same signs of operation are used as in arithmetic.

Any combination of numerals, letters, and signs of operation, used for the purpose of representing numbers, is called an *algebraic expression.*

E.g. 38, $18\,r$, $5\,n + 8\,n$, $3\,a - 2$ are algebraic expressions.

Expressions such as $5\,n + 8\,n$ and $3\,a - 2$ are said to be **written in symbols**; that is, by means of numerals, letters, and signs of operation.

The expression $3\,a - 2$ written in words would be "three times the number a diminished by 2." See how much simpler the algebraic expression is.

ORAL EXERCISES

Read the expressions in Examples 1 to 7.

1. $3 + 5$; $a + b$; $3\,k + 2\,n$; $2\,k + n$; $k + 2\,n$.
2. $7 - 3$; $a - b$; $3\,k - 2\,n$; $2\,k - n$; $k - 2\,n$.
3. 4×7; $3 \cdot 5$; ab; abc; $4\,mn$; $2\,ax$; $3\,cd$.
4. $10 \div 5$; $\frac{8}{3}$; $c \div d$; $m \div n$; $2\,k \div a$; $3\,n \div k$.
5. $3\,a$; $4\,abc$; $4 \cdot 5 \cdot 6$; $3\,mn$; $3\,m - 2\,n$; $4\,ab - 2\,c$.
6. $2\,k$; $2\,k + 1$; $2\,k - 1$; $4\,k + 1$; $4\,k - 1$.
7. $\frac{2}{3}\,h$; $\frac{3}{4}\,v$; $\frac{1}{5}\,n$; $\frac{1}{8}\,k$; $\frac{2}{5}\,abc$; $2\,ab$; $3\,cd - \frac{2}{3}\,mn$.
8. If one yard of wire costs $3\,¢$, how many cents will 5 yards cost? 8 yards? n yards? State the process in words.
9. If one carfare costs 5 cents, how much will 10 carfares cost? n carfares?
10. How many days in 3 weeks? in n weeks?
11. How many feet in 4 yards? in n yards?
12. How many inches in 12 feet? in 20 feet? in k feet?
13. How many cents in \$1? in \$5? in \$ n?

WRITTEN EXERCISES

1. If a and b are numbers, express in symbols their sum and also their product.

2. If m and n are numbers, write in symbols m divided by n; also m minus n.

3. If p and q are numbers, write the sum of 5 times p and 3 times q.

4. If a, b, and c are numbers, write in symbols that a multiplied by b equals c; also that c divided by a equals b.

If $a = 3$, $b = 5$, $c = 7$, find the value of each of the following algebraic expressions.

5. $a + b + c$.	10. $4c - 3a$.	15. $3b - 2c + 3$.
6. $a + b - c$.	11. $bc + ac$.	16. $4c - b + 3a$.
7. $2a - b + c$.	12. $ab - c + 2$.	17. $abc - a - b$.
8. $5b - 2c$.	13. $4a - 2b + 3c$.	18. $ab + bc + ac$.
9. $16a - 6c$.	14. $2a + ab + 7c$.	19. $ac + bc - ab$.

20. If m, n, and p are numbers, write 5 times the product of m and n, minus 3 times the product of m and p.

HISTORICAL NOTE

Origin of the Symbols of Operation. It was about the year 1500 A.D. that our present symbols indicating addition and subtraction first appeared in a book by a German named Johann Widmann. The sign $\times$ for multiplication was first used about 50 years later by an Englishman, William Oughtred. About the same time the sign $=$ was first used by Robert Recorde, also an Englishman; but the sign $\div$ for division does not appear until 1659, when it was used by a German, J. H. Rahn. It should not be imagined, however, that any one of these symbols came suddenly into general use. A distinguished historian of mathematics has said:

"Our present notation has arisen by almost insensible gradations as convenience suggested different marks of abbreviation to different authors; and that perfect symbolic language which addresses itself solely to the eye, and enables us to take in at a glance the most complicated relations of quantity, is the result of a long series of small improvements."

5. Factors. Many algebraic expressions, such as abc, $2mn$, $3xy$, represent products. Numbers which multiplied together form a product are called *factors of that product.*

E.g. 3 and 4 are factors of 12, as are also 2 and 6, 1 and 12. a, b, c, are factors of abc, as are also a and bc, b and ac, c and ab.

Coefficients. If an expression is the product of two factors, then either of these factors is called the *coefficient* of the other.

E.g. In $9rt$, 9 is the coefficient of rt, r is the coefficient of $9t$, and t is the coefficient of $9r$. But in such an expression the factor represented by Arabic numerals is usually regarded as the coefficient.

ADDITION AND SUBTRACTION OF NUMBERS HAVING A COMMON FACTOR

6. Addition. Two numbers which have a *common factor,* such as $5 \cdot 3$ and $8 \cdot 3$, may be added in two ways.

By the first method, *we perform the indicated multiplications and then add the products.*

Thus $5 \cdot 3 + 8 \cdot 3 = 15 + 24 = 39.$

By the second method, *we add the coefficients of the common factor and then multiply this sum by the common factor.*

Thus, $5 \cdot 3 + 8 \cdot 3 = 13 \cdot 3 = 39.$

This is evident, since 5 times a number and 8 times the *same* number make 13 times that number.

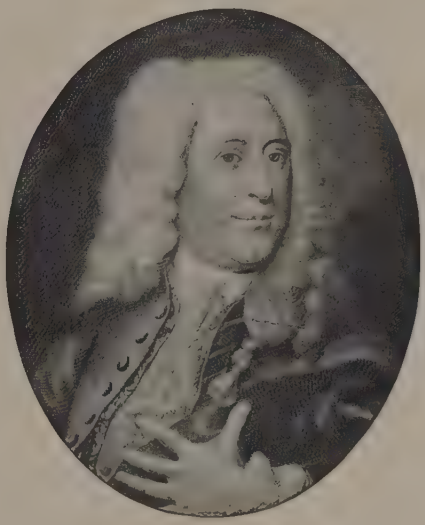
In case the common factor is represented by a letter, the second process only is available.

Thus $5n + 8n = 13n.$

ORAL EXERCISES

In the manner shown above add the following.

1. $2x + 4x.$
2. $4a + 3a.$
3. $13n + 4n.$
4. $3a + 2a.$
5. $5 \cdot 6 + 3 \cdot 6.$
6. $7b + 8b + b.$
7. $8t + 7t + 5t.$
8. $3r + 5r + 7r.$
9. $5 \cdot 9 + 4 \cdot 9 + 6 \cdot 9.$



Johann Widmann was born at Eger in Austria in 1460. He matriculated at Leipsic in 1480 and later became a lecturer there on algebra. In 1489 he wrote a treatise on mercantile arithmetic, in which are first found the symbols + and −, used as marks of excess and deficiency.

Widmann is chiefly interesting to us as the first German scholar whose name is associated with the subject of algebra. He has been called the originator of German algebra, though Christoff Rudolff, a pupil of Widmann's, was the first to write a textbook of algebra in the German language. Widmann was the author of a book on geometry.

7. Subtraction. Just as we add two numbers having a common factor by adding the coefficients of the common factor, so we subtract two such numbers by subtracting the coefficients.

Thus, from $64 = 8 \cdot 8$	From $84 = 12 \cdot 7$	From $17\,n$
subtract $48 = 6 \cdot 8$	subtract $49 = 7 \cdot 7$	subtract $6\,n$
Remainder $16 = 2 \cdot 8$	$35 = 5 \cdot 7$	$11\,n$

ORAL EXERCISES

In this way perform the following subtractions:

1. $8 \cdot 7 - 3 \cdot 7$.
2. $6 \cdot 99 - 5 \cdot 99$.
3. $6\,n - 2\,n$.
4. $8\,a - 3\,a$.
5. $10\,b - 4\,b$.
6. $7\,a - 4\,a$.
7. $23\,x - 16\,x$.
8. $15\,n - 3\,n$.
9. $14\,a - 8\,a$.
10. $12\,b - 9\,b$.
11. $8\,t - 2\,t$.
12. $19\,r - 11\,r$.

The foregoing examples illustrate

Learn.

Principle I

8. Rule. *To find the* **sum** *of two numbers having a common factor,* **add** *the coefficients of the common factor and multiply the result by the common factor.*

To find the **difference** *between two numbers having a common factor,* **subtract** *the coefficients of the common factor and multiply the result by the common factor.*

ORAL EXERCISES

Perform the following indicated operations:

1. $5\,x + 7\,x$.
2. $8\,x - 3\,x$.
3. $9\,a + 8\,a$.
4. $2\,m + 6\,m$.
5. $8\,y + 5\,y - 7\,y$.
6. $3\,a + 7\,a - 6\,a$.
7. $5\,ab + 7\,ab - 3\,ab$.
8. $3\,x + 9\,x - 8\,x$.
9. $4\,xy + 6\,xy + 4\,xy$.
10. $2\,ax + 3\,ax - 4\,ax$.
11. $6\,x + 14\,x - 7\,x$.
12. $20\,x - 8\,x - 3\,x$.
13. $5\,by + 7\,by - 3\,by$.
14. $4\,abc + 3\,abc + 2\,abc$.
15. $6\,ay + 9\,ay - 3\,ay$.
16. $12\,x + 8\,x + 4\,x$.

9. Checking Results. The substitution of numerical values for letters in order to test the correctness of an operation is called *checking the result.*

E.g. To check $2x + 7x + 4x = 13x$, we may substitute 2 for x.
Thus, $2 \cdot 2 + 7 \cdot 2 + 4 \cdot 2 = 4 + 14 + 8 = 26 = 13 \cdot 2$.

WRITTEN EXERCISES

Perform the following indicated operations and check the results in the first eight by letting $t = 1$, $n = 2$, $x = 1$, $k = 1$, $s = 1$, $a = 1$.

1. $68t - 11t$.
2. $15n + 25n - 18n$.
3. $70x - 15x + 7x - 23x$.
4. $48k - 3k - 2k + 6k$.
5. $20n - 16n + 2n$.
6. $25t + 20t - 3t$.
7. $28s - 3s + 20s$.
8. $36a - 14a + 3a - 2a$.
9. $7ab - 3ab + 2ab$.
10. $34rs - 12rs - 5rs$.
11. $34xy - 18xy - 4xy$.
12. $18mn - 7mn - 3mn$.
13. $17st + 3st - 12st$.
14. $12abc - 2abc - 6abc$.
15. $42xy + 6xy - 35xy$.
16. $29rst - 18rst - 6rst$.
17. $32ac - 17ac + 2ac$.
18. $91a - 81a + 2a$.
19. $16x + 24x + 8x - 40x$
20. $5y + 31y - 9y - 21y$.
21. $63c - 47c - 8c + 7c$.
22. $16t - 11t - 2t + 3t$.
23. $12xy - 9xy + 8xy$.
24. $39ab - 27ab - 8ab$.
25. $3rst + 9rst + 26rst - 18rst$.
26. $47abc - 14abc - 8abc + 3abc$.
27. $31xyz + 42xyz - 39xyz + 7xyz + 17xyz$.
28. $3mn + 24mn - 14mn + 8mn + 11mn$.
29. $78qr - 13qr + 8qr + 7qr - 12qr$.
30. $12lm + 7lm - 9lm + 46lm - 7lm$.
31. $145x + 17x - 125x + 280x$.
32. $40kl + 260kl - 34kl - 79kl$.
33. $7y + 14y + 28y + 56y + 112y$.
34. $93z + 47z + 82z + 235z - 406z$.

10. Equations and Identities. Two algebraic expressions representing the same number and connected by the sign $=$ form an *equation*, as $n + 3 = 5$.

The expressions thus connected are called the **members** of the equation. In the above equation, $n + 3$ is the **left member**, and 5 is the **right member**.

In general, an equation is true only when the letters have certain special values. Thus $n+3=5$ is true only when $n=2$.

The equation $5\,n + 8\,n = 13\,n$ is true for all values of n. Such an equation is called an *identity*.

When it is desired to emphasize that an equation is an identity, the sign $\equiv$ is used. That is, $3\,a + 5\,a \equiv 8\,a$.

An equation such as $3 \times 4 = 12$, in which both members are expressed in Arabic numerals is also called an *identity*.

11. Symbols of Aggregation. **Parentheses** are used to indicate that some operation is to be extended over the whole expression inclosed by them. Thus $2(x+y)$ means that the *sum* of x and y is to be multiplied by 2, while $2\,x+y$ means that x alone is to be multiplied by 2.

Instead of parentheses, **brackets** [ ], **braces** { }, or the **vinculum** $\overline{\quad}$, may be used with the same meaning. All such symbols are called *symbols of aggregation*.

E.g. $2(x+y)$, $2[x+y]$, $2\{x+y\}$, $2\,\overline{x+y}$ all mean the same thing.

ORAL EXERCISES

Evaluate (find the value of) each of the following when

$$a = 4,\ b = 2,\ c = 1,\ x = 4,\ y = 6.$$

1. $3(a+b)$.
2. $4(a-b)$.
3. $(x+y) \div 2$.
4. $(x+y) \div 2$.
5. $4\,a - b$.
6. $3\,a + b$.
7. $2[x+y-3]$.
8. $3\{a+b+c\}$.
9. $3(a+b)+c$.

Historical Note. Parentheses ( ) were first used with their present meaning by an Englishman, A. Girard, in a book on "Arithmetic," published in the year 1629. Brackets and braces are of later origin, as is also the sign $\equiv$ to denote identity.

MULTIPLICATION OF THE SUM OR DIFFERENCE OF TWO NUMBERS

12. Multiplying a Sum. The sum of two or more arithmetical numbers may be multiplied by another number in two ways.

By the first method *we add the numbers and then multiply their sum by the other number.*

Thus $4(2+7) = 4 \cdot 9 = 36$;
$3(3+8+9) = 3 \cdot 20 = 60.$

By the second method *we multiply each of the numbers separately and then add the products.*

Thus $4(2+7) = 4 \cdot 2 + 4 \cdot 7 = 8 + 28 = 36$;
$3(3+8+9) = 3 \cdot 3 + 3 \cdot 8 + 3 \cdot 9 = 9 + 24 + 27 = 60.$

But in case the numbers are represented by letters, the second process only is available.

E.g. $3(a+b) = 3a + 3b$ and $m(r+s) = mr + ms$.

ORAL EXERCISES

Multiply each of the following in two ways when possible:

1. $3(2+4)$.
2. $4(3+5)$.
3. $2(4+6)$.
4. $5(3+2)$.
5. $4(7+3)$.
6. $8(5+1)$.
7. $3(a+6)$.
8. $11(h+k)$.
9. $4(a+b+c)$.
10. $7(x+y)$.
11. $5(x+y+3)$.
12. $4(a+b+1)$.
13. $7(4+y+3)$.
14. $6(2+a+b)$.
15. $3(2+3+4)$.
16. $4(a+b+6)$.
17. $4(6+3)+2$.
18. $2(6+b)+3$.

13. Multiplying a Difference. The difference of two arithmetical numbers may likewise be multiplied by a given number in either of two ways.

E.g. $6(8-3) = 6 \cdot 5 = 30,$
or $6(8-3) = (6 \cdot 8) - (6 \cdot 3) = 48 - 18 = 30.$

But in case the numbers are represented by letters, the second process only is available.

E.g. $6(r-t) = 6r - 6t$ and $a(c-d) = ac - ad.$

ORAL EXERCISES

Perform as many as possible of the following multiplications in two ways:

1. $7(5-2)$.
2. $12(7-4)$.
3. $5(10-8)$.
4. $7(6-3)$.
5. $9(a-2)$.
6. $8(h-4)$.
7. $5(x-1)$.
8. $3(y-2)$.
9. $a(c-d)$.
10. $m(r-s)$.
11. $x(y-z)$.
12. $t(u-v)$.
13. $x(y-8)$.
14. $a(x-7)$.
15. $b(z-4)$.

The foregoing examples illustrate

Principle II

14. Rule. *To multiply the* **sum** *of two numbers by a given number, multiply each of the numbers separately by the given number, and* **add** *the products.*

To multiply the **difference** *of two numbers by a given number, multiply each of the numbers separately by the given number, and* **subtract** *the products.*

WRITTEN EXERCISES

1. Multiply $5+7+11$ by 3 without first adding, and then check by performing the addition before multiplying.

2. Multiply $m+n$ by 4 and check for $m=5$, $n=7$.

Solution. $4(m+n)=4m+4n$.

Check. $4(5+7)=4\cdot 12=48$, also $4\cdot 5+4\cdot 7=20+28=48$.

3. Multiply $x+y$ by r and check for $x=2$, $y=4$, $r=6$.

4. Multiply $r+s$ by k and check for $r=4$, $s=5$, $k=6$.

5. Multiply $a+b$ by m and check for $a=3$, $b=2$, $m=4$.

6. Multiply $m-n+2$ by c and check for $m=5$, $n=2$, and $c=3$.

7. Multiply $a-b-c$ by d and check for $a=10$, $b=3$, $c=4$, and $d=8$.

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DIVISION OF THE SUM OR DIFFERENCE OF TWO NUMBERS

15. Dividing a Sum or a Difference. In dividing the sum or difference of two arithmetical numbers by a given number, the process may be carried out in two ways.

By the first method:

$$(12 + 8) \div 2 = 20 \div 2 = 10;$$
$$(20 - 12) \div 4 = 8 \div 4 = 2.$$

By the second method:

$$(12 + 8) \div 2 = (12 \div 2) + (8 \div 2) = 6 + 4 = 10;$$
$$(20 - 12) \div 4 = (20 \div 4) - (12 \div 4) = 5 - 3 = 2.$$

If the numbers in the dividend are represented by letters, then usually the second method only is available.

E.g. $(r + t) \div 5 = (r \div 5) + (t \div 5)$.

Division in algebra is often indicated by the fractional form.

Thus, $(r + t) \div 5 = (r \div 5) + (t \div 5)$ is commonly written

$$\frac{r + t}{5} = \frac{r}{5} + \frac{t}{5}.$$

In either case this is read: *r plus t divided by 5 equals r divided by 5 plus t divided by 5.*

ORAL EXERCISES

Perform each of the following divisions in two ways when this is possible:

1. $(4 + 12) \div 4$.	6. $(12 - 6) \div 3$.	11. $(x + y) \div 4$.
2. $(8 + 4) \div 2$.	7. $(18 - 12) \div 6$.	12. $(m + r) \div 9$.
3. $(10 + 5) \div 5$.	8. $(21 - 14) \div 7$.	13. $(m - r) \div 9$.
4. $(9 + 12) \div 3$.	9. $(x - y) \div 2$.	14. $(r - t) \div 5$.
5. $(16 + 8) \div 4$.	10. $(y - z) \div 6$.	15. $(r + t) \div 5$.

16. $(8 + 6 - 4) \div 2$.
17. $(9 + 6 - 3) \div 3$.
18. $(12 - 4 + 8) \div 4$.
19. $(a + b - c) \div 2$.
20. $(a + 4 + 6) \div 2$.
21. $(8 - b - c) \div 4$.
22. $(9 + c + d) \div 3$.
23. $(x - y + 12) \div 6$.

The foregoing examples illustrate

Principle III

16. Rule. *To divide the* **sum** *of two numbers by a given number, divide each number separately and* **add** *the quotients.*

To divide the **difference** *of two numbers by a given number, divide each number separately and* **subtract** *the quotients.*

WRITTEN EXERCISES

1. Divide $72 + 56$ by 8 without first adding.
2. Divide $144 - 36$ by 12 without first subtracting.
3. Divide $r + t$ by 5 and check the quotient when $r = 15$, $t = 25$; also when $r = 60$, $t = 30$.
4. Multiply $7 + 9$ by 3 without first adding 7 and 9.
5. Find the product of 12 and $a + b$, checking the result when $a = 5$, $b = 7$.

Perform the following indicated operations:

6. $3(a + b + c + d)$. Check for $a = 1$, $b = 2$, $c = 3$, $d = 4$.
7. $7(r - s + t - x)$. Check for $r = t = 5$, $s = x = 4$.
8. $(m + n + r) \div 4$. Check for $m = 64$, $n = 32$, $r = 8$.
9. $(x + y + z) \div 5$. Check for $x = 15$, $y = 10$, and $z = 5$.
10. $74\,rs - 67\,rs - 2\,rs - 3\,rs$.
11. $(63 - 35 - 14 + 21) \div 7$.
12. $a(4 - d + b + c + 3)$.
13. $(21 - x - y + 3 + c)k$.

HISTORICAL NOTE

Fundamental Laws for Multiplication and Division. The fundamental character of Principles II and III was not fully appreciated until the first part of the last century. Principle II states what is called the **Distributive Law of Multiplication** with respect to addition and subtraction. That is, the multiplier is *distributed* over the multiplicand. The name was first used by a Frenchman, F. J. Servois, in a paper published in 1814. Principle III states the **Distributive Law of Division** with respect to addition and subtraction. Compare notes on pages 69 and 84.

MISCELLANEOUS EXERCISES

1. $(48 + 36 + 24 + 12) \div 6$.
2. $(35 - 20 + 30 - 40) \div 5$.
3. $3(x + y + z) + 2(y - z + x)$.
4. $3(a + b + c) + 2(a - b + c)$.
5. $5(x + y + z) + 3(x - y + z)$.
6. $4(ab + cd + ef) + 2(ab + cd - ef)$.
7. $49\,pq + 18\,pq - 62\,pq + 3\,pq$.
8. $13\,xyz + 3\,xyz - 8\,xyz - 7\,xyz$.
9. $(q + r + s + t - a - b) \div c$.
10. $35\,lm - 33\,lm + 7\,lm - 3\,lm - 2\,lm$.
11. $k(l + m + n + r - s - t)$.
12. $a(c + d - e + f - g)$.
13. $27\,abc - 19\,abc - 4\,abc + 8\,abc$.
14. $(a + r + s - t - q) \div 3$.

15. For what values of a, b, c, d are the following equations true?

(a) $ab + ac + ad = a(b + c + d)$.

(b) $ab + ac - ad = a(b + c - d)$.

(c) $\dfrac{a + b + c}{d} = \dfrac{a}{d} + \dfrac{b}{d} + \dfrac{c}{d}$.

MULTIPLICATION OF A PRODUCT

17. Multiplying a Product. In multiplying a product like $3 \cdot 5$ by 2 the work may be carried out in three different ways.

Thus, $2 \cdot (3 \cdot 5) = 2 \cdot 15 = 30$;
or $2 \cdot (3 \cdot 5) = 6 \cdot 5 = 30$;
or $2 \cdot (3 \cdot 5) = 3 \cdot 10 = 30$.

In multiplying $3\,n$ by 2 we could *indicate* the work in these three ways, but we can *perform* it in the second way only; namely, $2 \cdot (3\,n) = 6\,n$.

We may evidently choose any *one* of the factors of the multiplicand and multiply this one alone by the multiplier.

ORAL EXERCISES

Find each product in several ways and verify by comparing results.

1. $2(2 \cdot 3)$.
2. $2(3 \cdot 4)$.
3. $2(4 \cdot 5)$.
4. $3(3 \cdot 4)$.
5. $2(2 \cdot 3 \cdot 4)$.
6. $3(2 \cdot 3 \cdot 4)$.
7. $2(5 \cdot 10)$.
8. $2(5 \cdot 6)$.
9. $4(5 \cdot 5)$.
10. $3(5 \cdot 4)$.
11. $8(6 \cdot 2)$.
12. $9(3 \cdot 4)$.
13. $7(2 \cdot 3)$.
14. $10(2 \cdot 4)$.
15. $2(4 \cdot 5 \cdot 2)$.

These examples illustrate

Principle IV

18. **Rule.** *To multiply the product of several factors by a given number, multiply any* **one** *of the factors by that number.*

Principles IV and II should be carefully contrasted.

Thus, $2(2 \cdot 3 \cdot 5) = 4 \cdot 3 \cdot 5 = 2 \cdot 6 \cdot 5 = 2 \cdot 3 \cdot 10 = 60$,
but $2(2 + 3 + 5) = 4 + 6 + 10 = 20$.

In multiplying the product of several numbers we operate upon any **one** *of them, but in multiplying the sum or difference of numbers we operate upon* **each** *of them.*

WRITTEN EXERCISES

Multiply as many as possible of the following in two or more ways.

1. $7(3 \cdot 4 \cdot 5)$.
2. $8(7 \cdot 2 \cdot 3)$.
3. $9(2 \cdot 3 \cdot 4)$.
4. $15(2\,ab)$.
5. $13(5\,xy)$.
6. $23(8\,xyz)$.
7. $4(19 \cdot 5)$.
8. $15(7\,abc)$.
9. $15(7\,ab)$.
10. $33(4\,mn)$.
11. $47(2\,xy)$.
12. $12(16\,rs)$.
13. $6(30\,xy)$.
14. $9(3 \cdot 4 \cdot 5)$.
15. $7(8 \cdot 5)$.
16. $8(9 \cdot 5)$.
17. $3(9 \cdot 2 \cdot 6)$.
18. $2(4 \cdot 7 \cdot 5)$.
19. $6(5 \cdot 4 \cdot 6)$.
20. $5(4 \cdot 5 \cdot x)$.
21. $4(5 \cdot 8 \cdot y)$.

DIVISION OF A PRODUCT

19. **Dividing a Product.** Just as in multiplying the product of several factors by a given number, so also in dividing such a product by a given number, we may operate upon any one of the factors separately.

E.g. $(4 \cdot 6 \cdot 10) \div 2 = 240 \div 2 = 120.$
Also $(4 \cdot 6 \cdot 10) \div 2 = 2 \cdot 6 \cdot 10 = 120,$
$(4 \cdot 6 \cdot 10) \div 2 = 4 \cdot 3 \cdot 10 = 120,$
and $(4 \cdot 6 \cdot 10) \div 2 = 4 \cdot 6 \cdot 5 = 120.$

Note that in each case only one factor is divided.

ORAL EXERCISES

Perform each of the following divisions in more than one way where possible:

1. $(5 \cdot 8 \cdot 3) \div 2.$
2. $20\,abc \div 4.$
3. $12\,abc \div 3.$
4. $(3 \cdot 20 \cdot 8) \div 4.$
5. $14\,xyz \div 7.$
6. $12\,abc \div c.$
7. $(10 \cdot 5 \cdot 3) \div 5.$
8. $14\,xyz \div x.$
9. $(16 \cdot 18 \cdot 13) \div 8.$
10. $5\,abc \div a.$
11. $15\,xy \div 3.$
12. $5\,(10\,x + 15\,y) \div 5.$
13. $8\,(3\,x - 4\,y) \div 4.$
14. $6\,(2\,s - 3\,t) \div 3.$

The foregoing examples illustrate

Principle V

20. **Rule.** *To divide the product of several factors by a given number divide any one of the factors by that number.*

21. **Cancellation.** Principle V is already known in arithmetic in the process called *cancellation.*

Thus, in the fraction $\frac{2 \cdot 6 \cdot 9}{3}$, the 3 may be cancelled out of either 6 or 9, giving $\frac{2 \cdot 6 \cdot 9}{3} = 2 \cdot 2 \cdot 9$ or $2 \cdot 6 \cdot 3$.

Contrast Principles III and V.

By Principle V, $(4 \cdot 6 \cdot 8) \div 2 = 2 \cdot 6 \cdot 8 = 4 \cdot 3 \cdot 8 = 4 \cdot 6 \cdot 4 = 96$.
By Principle III, $(4 + 6 + 8) \div 2 = 2 + 3 + 4 = 9$.

That is, *in dividing the product of several numbers we operate upon any* **one** *of them, but in dividing their sum or difference we operate upon* **each** *of them.*

WRITTEN EXERCISES

Perform the following indicated operations, remembering that a fraction is merely an indicated division:

1. $3a(7 - c + b) \div a$.
2. $5bc(d - e + 3) \div bc$.
3. $19(3a - 6b + 9c) \div 3$.
4. $13(8 - 4b + 12a) \div 4$.
5. $14(7 - 7m + 14n) \div 7$.
6. $12a(3b - 3c + 9) \div 3$.
7. $24(16b - 8c + 24d) \div 8$.
8. Divide $7a \cdot 14b \cdot 21c$ by 7 in three different ways.
9. Add $5a$, $\frac{12a}{2}$, $\frac{21a}{3}$, and $\frac{18a}{6}$, using Principles V and I
10. From $\frac{28xy}{4}$ subtract $\frac{21xy}{7}$, using Principles V and I.
11. From $\frac{14a}{2} + \frac{10a}{5}$ subtract $\frac{6a}{3}$.
12. Find the sum of $\frac{16x}{8}$, $\frac{20x}{5}$, $\frac{16x}{4}$, $7x$, and $3x$.
13. Find the sum of $\frac{100rs}{10}$, $\frac{90rs}{9}$, and $\frac{25rs}{5}$.
14. From $25xy$ subtract $\frac{13xyz}{z}$.
15. Add $\frac{8abc}{c} + \frac{18ab}{3} + \frac{7abd}{d} + \frac{abe}{e}$.
16. Add $\frac{18a}{6} + \frac{5ab}{b} + \frac{4ax}{x} + \frac{am}{m}$.
17. Add $\frac{56x}{28} + \frac{48x}{12} + \frac{32x}{4} + \frac{24x}{3}$.

ORDER OF INDICATED OPERATIONS

22. Order of Operations. In a succession of indicated operations the final result depends in some cases upon the *order* in which the operations are performed, while in some cases it does not. This is shown in the following examples:

(1) $6 + 3 \cdot 8$ would give $9 \cdot 8 = 72$, if the addition were performed first; and would give $6 \div 24 = 30$, if the multiplication were performed first.

(2) $24 \div 2 \cdot 3$ gives $12 \cdot 3 = 36$, if the division is performed first; and gives $24 \div 6 = 4$, if the multiplication is performed first.

(3) $24 \div 6 \div 2$ equals 2 if we first divide 24 by 6, and equals 8 if we first divide 6 by 2.

(4) $12 \times 6 \div 3 = 24$ no matter in what order the operations are performed.

(5) $18 + 10 - 8 = 20$, no matter in what order the operations are performed.

(6) $4 \times 3 \times 2 = 24$, no matter in what order we multiply.

23. To prevent mistakes, and to make usage uniform, the following rules have been adopted:

In an expression involving additions, subtractions, multiplications, and divisions, when no symbols of aggregation (§ 11) *are involved,*

(1) *All multiplications are performed first, and these may be taken in any order;*

(2) *All divisions are performed next, and these are taken in the order in which they occur from left to right;*

(3) *Finally, additions and subtractions are performed, and these may be taken in any order.*

These rules are illustrated by the following examples:

(1) $5 + 3 \cdot 4 - 8 \div 2 = 5 + 12 - 8 \div 2 = 5 + 12 - 4 = 13$;

(2) $8 - 2 \div 2 - 2 \times 2 + 3 = 8 - 2 \div 2 - 4 + 3 = 8 - 1 - 4 + 3 = 6$;

(3) $8 \cdot 2 \div 4 + 28 \div 2 \cdot 2 = 16 \div 4 + 28 \div 4 = 4 + 7 = 11$;

(4) $16 + 24 \div 3 \cdot 4 - 8 \cdot 3 \div 2 = 16 + 24 \div 12 - 24 \div 2$
$= 16 + 2 - 12 = 6$.

ORAL EXERCISES

1. What is the value of $4 \cdot 3 + 2 \cdot 4 - 3 \cdot 5$?
2. What is the value of $5 \cdot 4 - 6 + 2 \cdot 5$?
3. What is the value of $2 \cdot 3 \cdot 4 + 12 \div 4$?
4. What is the value of $3(4 + 2) + 6$?
5. What is the value of $2(3 + 2 \cdot 3) - 4(1 + 3)$?

If $a = 2$, $b = 3$, $c = 4$, find the value of each of the following. Remember that a fraction is an indicated division.

6. $2ab - c$.
7. $\dfrac{bc - 4a}{a}$.
8. $\dfrac{3ac + 2b}{2b}$.
9. $5a + 2b - 3c$.
10. $10c - 10b - 4a$.
11. $9a + 3c - 10b$.
12. $\dfrac{8a + c + 5b}{5}$.
13. $\dfrac{4c - 2b + 3a}{4a}$.
14. $2a + 3b + 4c$.
15. $6a - b - c$.

WRITTEN EXERCISES

If $a = 1$, $b = 2$, $c = 3$, $x = 4$, $y = 5$, $z = 6$, find the value of each of the following expressions:

16. $(6ab \div 4) \div 3$.
17. $6ab \div (4 \div 3)$.
18. $(12cx \div 2z)b$.
19. $12cbx \div 2z$.
20. $12cx \div 2zb$.
21. $(6xyz \div 2b) \div 2c$.
22. $6xyz \div (3b \div 2c)$.
23. $5abc - 3x + 2yz$.
24. $3xyz - 8a + 5b$.
25. $3xyz - (8a + 5b.)$
26. $3axy - 4c - b$.
27. $3axy - 4(c - b)$.
28. $3axy - 4(c + b)$.
29. $2bz - (2x - c)$.
30. $2bz - 2(x - c)$.
31. $2bz - 2x - c$.
32. $30 - (x - a)$.
33. $30 - x - a$.
34. $3 + 5 \cdot 6 - 4 \cdot 3$.
35. $3 + 5(6 - 4) \cdot 3$.
36. $5(a + b + c - x)$.
37. $2cxy \div ab + 5$.
38. $4xy \div 3ab + 4$.
39. $4xy \div (4a \div b)$.
40. $2axy \div (5b \div 2a)$.
41. $(2axy \div 5b) \div 2a$.

24. Importance of the Principles. The five principles studied in this chapter, together with others which will be introduced when needed, will be found of increasing importance as we proceed. Your success in the further study of algebra will depend in no small degree upon the clearness with which you understand their real significance.

The most effective way to master these principles is by means of simple numerical examples such as were used in introducing each one. Make a list of these principles in abbreviated form for yourself and note how frequently your own errors and those of your classmates are due to direct violations of one or more of them.

WRITTEN EXERCISES

If $a = 5$, $b = 3$, $c = 2$, find the value of each of the following:

1. $ac + bc$ and $(a + b)c$.
2. $ac - bc$ and $(a - b)c$.
3. abc, $a \times (bc)$ and $b \times (ac)$.
4. $a(b + c)$ and $ab + ac$.
5. $a(b - c)$ and $ab - ac$.
6. $\frac{ab}{c}$, $\frac{a}{c} \times b$ and $a \times \frac{b}{c}$.
7. $\frac{a + b}{c}$ and $\frac{a}{c} + \frac{b}{c}$.
8. $\frac{a - b}{c}$ and $\frac{a}{c} - \frac{b}{c}$.

9. In each of the above compare the results obtained in the two parts.

10. $\frac{4a - 5b}{5} + \frac{b + 9}{4}$
11. $\frac{8b - 9c}{3} + \frac{3a + b}{6}$
12. $\frac{8ab - 5bc}{6} + \frac{2ab + 3c}{b}$
13. $\frac{6ac + 5bc}{2a} + \frac{ab + 3c}{b}$
14. $\frac{12a + 5b}{5} + \frac{6c + 4b}{6}$
15. $\frac{12bc + 4b}{12} + \frac{3ac + 2b}{6}$
16. $\frac{10ab - 3c}{2b} + \frac{5bc - 2a}{a}$
17. $\frac{4b + 6a - 3c}{6}$
18. $\frac{12c + 4ac - 4c}{8}$
19. $(6a + 5c) + 2a$

REVIEW QUESTIONS

1. How would $3 \cdot 5$ and $7 \cdot 5$ be added in arithmetic? Why cannot $3\,n$ and $7\,n$ be added in the same manner? State the principle by which $3\,n$ and $7\,n$ are added. Test the identity $3\,n + 7\,n = 10\,n$ by substituting any convenient value for n.

2. What kind of numbers may be added by Principle I? Have the numbers ac and bc a common factor? What is it? What is the coefficient of this common factor in each? What is the sum of these coefficients? Is the equality $ac + bc = (a + b)\,c$ true no matter what numbers are represented by a, b, and c? When this can be said of an equality, what is it called?

3. How is $5 \cdot 9$ subtracted from $11 \cdot 9$ in arithmetic? In what different manner may this operation be performed? Why is it sometimes necessary to perform subtraction in the second way? In the identity $31\,x - 12\,x = 19\,x$, what number is represented by x? Test the equality by substituting any convenient number for x. Is this equality true for every value of x?

Principle I may be conveniently abbreviated as follows:

$$\begin{cases} ac + bc = (a+b)\,c, \\ ac - bc = (a-b)\,c. \end{cases}$$

4. How is $11 + 3$ multiplied by 4 in arithmetic? In what different way may this operation be performed? Why is it sometimes necessary to multiply in the second way? State in full the principle by which $a + 8$ is multiplied by 7.

Principle II may be abbreviated thus:

$$\begin{cases} c(a + b) = ca + cb, \\ c(a - b) = ca - cb. \end{cases}$$

Notice that the identities in Principle II are the same as those in Principle I read in reverse order. Thus, Principle II may be called the reverse or converse of Principle I.

5. How is $12 + 18$ divided by 6 in arithmetic? In what different way may this division be performed? Why is it sometimes necessary to perform division in the second way? State in full the principle used in performing the operation $(6x + 9y) \div 3$. How do you divide $(6x - 9y)$ by 3?

Principle III may be abbreviated thus:

$$\frac{a+b}{k} = \frac{a}{k} + \frac{b}{k}, \qquad \frac{a-b}{k} = \frac{a}{k} - \frac{b}{k}.$$

6. How is the product $2 \cdot 3 \cdot 5$ multiplied by 4 in arithmetic? In what different way may this multiplication be performed? Why is it ever performed in the second way? What are the factors of ab? How is the product of two numbers multiplied by another number? Should *both* factors be multiplied by the number, or only one? Is it permissible to multiply *either one* we choose?

Principle IV is abbreviated thus:

$$k \times (ab) = (ka) \times b = a \times (kb).$$

7. Divide $2 \cdot 4 \cdot 6 \cdot 20$ by 2 without first performing the multiplication indicated in $2 \cdot 4 \cdot 6 \cdot 20$. Do this in several ways and show that all the quotients obtained are equal. State in full the principle used.

Principle V is abbreviated thus:

$$(ab) \div k = \frac{a}{k} \times b = a \times \frac{b}{k}.$$

8. Contrast Principles II and IV; also III and V.

9. Why are our numerals called Arabic numerals? When and how were they brought into Europe?

10. What is the most important point in which the Arabic system differs from the Roman system?

11. What is meant by the distributive law of multiplication and the distributive law of division?

CHAPTER II

EQUATIONS AND PROBLEMS

The principles developed in the last chapter will now be used in the solution of equations and problems.

SOLUTION OF EQUATIONS

25. Unknowns. A letter whose value we seek by means of an equation is called an *unknown* in that equation.

Satisfying an Equation. A number is said to *satisfy an equation* if, when substituted for the unknown, it reduces the equation to an identity.

Thus, $3x = 18$ is said to be satisfied by $x = 6$, because this value of x reduces the equation to the identity $18 = 18$.

Equivalent Equations. If two equations which are not identities are satisfied by the same numbers, they are called *equivalent equations.*

Thus, $3x = 18$ and $6x = 36$ are equivalent equations because they are not identities and are both satisfied by $x = 6$.

Also, $n + 2 = 5$ and $2n + 4 = 10$ are equivalent equations, because they are both satisfied by $n = 3$ and by no other value of n.

ORAL EXERCISES

1. Is $x + 4 = 9$ satisfied by $x = 4$? by $x = 5$? by $x = 6$?
2. Is $2x + 9 = x + 12$ satisfied by $x = 2$? by $x = 3$?
3. Is $\frac{x}{4} + 3 = 6$ satisfied by $x = 4$? by $x = 8$? by $x = 12$?
4. Is $3(x + 2) = 3x + 6$ satisfied by $x = 1$? by $x = 2$? by $x = 3$? by $x = 4$?

26. **To solve an equation** is to find a value of the unknown which will satisfy the equation.

A value of the unknown which satisfies an equation is called a **root** or **solution** of the equation.

The following examples illustrate methods used in solving equations.

Example 1. Solve the equation $x - 5 = 9$. (1)

Solution. This equation states that 9 is 5 less than the number x. That is, if 9 be increased by 5, the result is x.

Hence, $x = 14$ is the solution of equation (1).

This result may be obtained by adding 5 *to each member of equation* (1), thus
$$x + 5 - 5 = 9 + 5,$$
or
$$x = 14. \quad (2)$$

Example 2. Solve the equation $x + 7 = 12$. (1)

Solution. This equation states that 12 is 7 more than the number x. That is, if 12 be diminished by 7, the result is x.

Hence, $x = 5$ is the solution of equation (1).

This result may be obtained by subtracting 7 *from both members of* (1), thus
$$x + 7 - 7 = 12 - 7,$$
or
$$x = 5. \quad (2)$$

Example 3. Solve the equation $\frac{1}{3}x = 7$. (1)

Solution. This equation states that one third of the number x is 7. That is, x is 3 times 7, or 21.

Hence $x = 21$ is the solution of equation (1).

This result may be obtained by multiplying both members of (1) *by* 3, thus
$$3 \times \tfrac{1}{3}x = 3 \cdot 7,$$
or
$$x = 21. \quad (2)$$

Example 4. Solve the equation $5x = 30$. (1)

Solution. This equation states that 5 times the number x is 30. That is, x is one fifth of 30, or 6.

Hence, $x = 6$ is the solution of the equation.

This result may be obtained by dividing both members of equation (1) *by* 5, thus
$$\frac{5x}{5} = \frac{30}{5},$$
or
$$x = 6. \quad (2)$$

ORAL EXERCISES

Solve the following equations and explain each step involved, as in the foregoing illustrative examples.

1. $2x = 16$.	7. $x + 4 = 8$.	13. $w + 2 = 14$.
2. $11x = 33$.	8. $x + 9 = 16$.	14. $w - 2 = 14$.
3. $\frac{1}{3}y = 6$.	9. $x - 2 = 8$.	15. $\frac{1}{4}w = 20$.
4. $\frac{1}{5}x = 7$.	10. $x - 3 = 10$.	16. $7w = 28$.
5. $\frac{1}{6}n = 20$.	11. $x + 8 = 16$.	17. $3n - 1 = 5$.
6. $6w = 42$.	12. $x - 3 = 12$.	18. $3n + 1 = 10$.

27. **Deriving Equivalent Equations.** The above examples illustrate four ways of operating upon an equation, so as to produce in each case a new equation *which is equivalent to the original equation.*

Each of these operations *changes the value of both members, but changes them both in the same way.*

The object of each step is to obtain an equation whose solution is more apparent than that of the preceding equation.

In the example below, equations (2) and (3) show two ways *of changing the form, but not the value, of one member alone,* thus producing a new equation which is equivalent to the original equation.

Example 5. Solve the equation

$$w + 2(w + 5) = 58. \qquad (1)$$

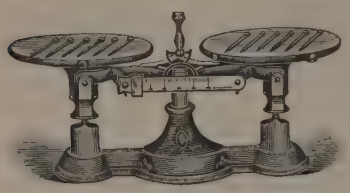
By Principle II,	$w + 2w + 10 = 58.$	(2)
By Principle I,	$3w + 10 = 58.$	(3)
Subtracting 10 from both members,	$3w = 48.$	(4)
Dividing both members by 3,	$w = 16.$	(5)

Check. Putting $w = 16$ in (1), $16 + 2(16 + 5) = 16 + 2 \cdot 21 = 58.$

The operations involved in passing from (1) to (2) and (2) to (3) are called **form changes.**

All the operations involved in Principles I to V are *form changes* of this character. See the list at the end of Chapter I. Other form changes will be considered as need arises.

28. **Illustrating the Operations on an Equation.** The members of an equation may be likened to the **scale-pans of a common balance** in which are placed objects of uniform weight, say tenpenny nails. The scales balance only when the weights are the same in both pans; that is, when the number of nails is the same.



ORAL EXERCISES

1. If there are ten nails on each side, do the scales balance?

2. If now six more nails are added to one side, what must be done to the other side to keep the scales balanced? If three nails are subtracted from one side what must be done to the other side to keep the scales balanced?

3. If the number of nails on one side is doubled, what must be done to the other side to keep the balance? If the number is divided by four?

4. If the scales are balanced, and if the nails in either pan are rearranged in any way, will the scales continue to balance?

Preserving the Balance. The above examples show that if the scales are in balance, they will remain so under two kinds of changes in the weights:

(*a*) When the number of nails in the two pans is increased or diminished by the same amount; corresponding to **like changes in value of both members of an equation.**

(*b*) When the number in each pan is left unaltered but the nails are rearranged in groups or piles in any manner; corresponding to **form changes** in either member of an equation.

The equation, then, is like a balance, and its members are to be operated upon only in such ways as to preserve the balance.

write + check.

WRITTEN EXERCISES

Solve the following equations as in example 5, § 27:

1. $x+4x=15$.
2. $12x+3x=30$.
3. $7x-3x=12$.
4. $x+3(x-1)=17$.
5. $4(3x-x)=8$.
6. $5(2x+1)=15$.
7. $3(4r-2)=18$.
8. $y+2(y+2)=16$.
9. $r+3(r+4)=24$.
10. $6l+2(l-3)=34$.
11. $\frac{1}{3}w+4w=26$.
12. $\frac{1}{2}w+2w+3=13$.

The foregoing examples illustrate

Principle VI

29. **Rule.** *An equation may be changed into an equivalent equation by means of any of the following operations:*

(1) *Adding the same number to both members;*

(2) *Subtracting the same number from both members;*

(3) *Multiplying both members by any known number not zero;*

(4) *Dividing both members by any known number not zero;*

(5) *Changing the form of either member in any way which leaves its value unaltered.*

The operations under Principle VI are hereafter referred to in detail by means of the initial letters, *A* for *addition*, *S* for *subtraction*, *M* for *multiplication*, *D* for *division*, and *F* for *form changes* which leave the value of a member unaltered.

Note. — The statements (1) to (4) in Principle VI include the following so-called **Axioms or self-evident truths**:

(1) If equals are added to equals, the sums are equal.

(2) If equals are subtracted from equals, the remainders are equal.

(3) If equals are multiplied by equals, the products are equal.

(4) If equals are divided by equals, the quotients are equal.

Principle VI, however, includes *more than these axioms*, since it states that the new equation obtained each time is *equivalent* to the original equation.

DIRECTIONS FOR WRITTEN WORK

30. In solving an equation the successive steps may be written as in the following examples:

1. $25(n+1)+6(4n-3)=50+31n+2(3-n)-9.$ (1)

By *F*, using Principle II, we obtain from (1)

$$25n+25+24n-18=50+31n+6-2n-9. \quad (2)$$

By *F*, using Principle I, we obtain from (2)

$$49n+7=29n+47. \quad (3)$$

Subtracting 7 and $29n$ from each member of (3) and using Principle I, we have

$$20n=40. \quad (4)$$

Dividing each member of (4) by 20,

$$n=2. \quad (5)$$

Check. Substitute $n=2$ in equation (1).

For convenience this work can be abbreviated as follows:

$$25(n+1)+6(4n-3)=50+31n+2(3-n)-9. \quad (1)$$

By *F*, II, $25n+25+24n-18=50+31n+6-2n-9.$ (2)

By *F*, I, $49n+7=29n+47.$ (3)

By *S* | 7, $29n$, $20n=40.$ (4)

By *D* | 20, $n=2.$ (5)

S | 7, $29n$ means that 7 and $29n$ were subtracted from both members of the preceding equation. *D* | 20 means that the members of the preceding equation were divided by 20.

Similarly, in case we wish to indicate that 6 is to be added to each member of an equation, we should write *A* | 6, and if each member is to be multiplied by 8, we should write *M* | 8.

2. $17n+4(2+n)-6=5(4+n)-5+3n.$ (1)

By *F*, II, $17n+8+4n-6=20+5n-5+3n.$ (2)

By *F*, I, $21n+2=15+8n.$ (3)

By *S* | 2, $8n$, $21n-8n=15-2.$ (4)

By *F*, I, $13n=13.$ (5)

By *D* | 13, $n=1.$ (6)

Check. Substitute $n=1$ in equation (1).

3. $\frac{x}{3}+\frac{x}{2}+x=11.$ (1)

By *M* | 6, $2x+3x+6x=66.$ (2)

By *F*, I, $11x=66.$ (3)

By *D* | 11, $x=6.$ (4)

31. Transposing Terms. By use of Principle VI, a term may be **transposed** from one member of an equation to the other, provided its sign is changed.

E.g. in deriving equation (4) from (3) in Example 2, page 30, $8n$ is subtracted from both sides by mentally *dropping* it on the right and *indicating* its subtraction on the left. Likewise when 2 is subtracted from both sides it *disappears* on the left and *appears* on the right with the *opposite* sign. Each of these indicated subtractions might have been performed *mentally*, thus writing equation (5) directly from (3).

After a little practice this shorter process of *transposing terms, changing the signs,* and *combining similar terms mentally* should always be used.

6 Ex 16

WRITTEN EXERCISES

Solve the following equations, putting the work in a form similar to that on page 30. Check the first ten.

1. $13x + 40 - x = 88.$
2. $3x + 15 + 2x = 18 + 4x.$
3. $5x + 3 - x = x + 18.$
4. $13y + 12 + 5y = 32 + 8y.$
5. $4m + 6m + 4 = 9m + 6.$
6. $7m + 18 + 3m = 2m + 50.$
7. $y + 42 + 45y = 76 + 12y.$
8. $42x + 56 = 20x + 122.$
9. $12m + 3 - 3m = 38 + 2m$
10. $15m + 3 - 2m = 3m + 53.$
11. $a + 7 + 3a = 2a + 45.$
12. $5b + 30 + 6b = 3b + 150.$
13. $3c + 18 + 14c = 6c + 51.$
14. $17x + 4 + 3x = 7x + 30.$
15. $3y + 4 + 2y + 6 = y + 7 + y + 3 + 30.$
16. $5x + 3 + 2x + 3 = 2x + 5 + 3x + 3 + x.$
17. $2x + 4x + 9 - x + 6 = 20 + 2x + 5 + x.$
18. $18 + 6m + 30 + 4m = 4m + 8 + 12 + 3m + 3 - m + 29.$
19. $6x + 8 + x + 4 + 5x = 7x + 32 - x - 20.$
20. $32x + 4 + 7x = 66 + 3x + 5x.$
21. $7(m + 6) + 10m = 42 + 5m + 24.$
22. $6x + 4(4x + 2) + 3(2x + 7) = 85.$
23. $8 + 7(6 + 6n) + 2n = 2(4n + 5) + 18n + 49.$
24. $5(9x + 3) + 4(3x + 2) = 18x + 36.$

32. **An equation may be translated into a problem.** For example, the equation $21x + 2 = 8x + 15$ may be interpreted as follows: *Find a number such that 21 times the number plus 2 is 15 greater than 8 times the number.*

ORAL EXERCISES

Translate each of the following into a problem:

1. $2x + 5 = 21$.
2. $3x - 7 = 19$.
3. $6 + 5x = 16$.
4. $4x - 3 = 2x + 9$.
5. $3(x + 1) = 18$.
6. $2x + 3 + x = x + 7$.
7. $5x - 3 - x = x + 9$.
8. $8 + 3x + 2x = 4x + 19$.

HISTORICAL NOTE

Origin of the Name Algebra. The Arabs brought their first algebra into Europe in the first half of the ninth century. It bore the name Al-jebr-wàl-muqābala. The word *al-gebr*, from which the word *algebra* is derived, means "transposition" and refers to transposing terms in solving equations. The word *wàl-muqābala* means the process of simplification or *form changes*. Thus it appears that the Arabs regarded the solution of equations as the main business of algebra.

SOLUTION OF PROBLEMS

33. One great object in the study of algebra is to **simplify the solution of problems.** This is done by using letters to represent numbers, by stating problems in the form of equations, and by the systematic solutions of the equations.

Illustrative Problem. 1. The shortest railway route from Chicago to New York is 912 miles. How long does it take a train averaging 38 miles an hour to make the journey?

Solution. Let t be the number of hours required. Then $38t$ is the distance traveled. But 912 miles is the given distance traveled.

Hence $38t = 912$. (1)

By $D \mid 38$ $t = 24$. (2)

Check. The distance traveled in 24 hours at 38 miles per hour is

$$38 \times 24 = 912.$$

Hence $t = 24$ satisfies the conditions of the problem.

Illustrative Problem. 2. For how many years must \$ 850 be invested at 5 % simple interest in order to yield \$ 255?

Solution. From arithmetic, we have

$$principal \times rate \times time = interest,$$

or
$$prt = i.$$

Hence, from the conditions of the problem,

$$850 \times .05 \times t = 255, \tag{1}$$

or
$$42.5\,t = 255.$$

By $D \mid 42.5$,
$$t = 6. \tag{2}$$

Check. The interest on \$ 850 for 6 years at 5 % is

$$6 \times .05 \times 850 = 255.$$

Hence $t = 6$ satisfies the conditions of the problem.

NOTE. — Substituting $t = 6$ in equation (1) would not fully check the solution, since that equation might be incorrect. It is necessary to see that the solution satisfies the *problem itself.*

Illustrative Problem. 3. A boy, an apprentice, and a master workman have the understanding that the apprentice shall receive twice as much as the boy and the master workman five times as much as the boy. How much does each get, if the total amount received for a piece of work is \$ 104?

Solution. Let n represent the number of dollars received by the boy.

Then, $2\,n$ is the number of dollars received by the apprentice,

and $5\,n$ is the number of dollars received by the master workman.

Hence, $n + 2\,n + 5\,n$ represents the total amount received.

Therefore,
$$n + 2\,n + 5\,n = 104. \tag{1}$$

By Principle I,
$$8\,n = 104. \tag{2}$$

By $D \mid 8$,
$$n = 13. \tag{3}$$

Hence, the amount received by the boy is 13 dollars. The apprentice receives $2\,n$ dollars or \$ 26, and the master workman receives $5\,n$ dollars or \$ 65.

Check. By the conditions of the problem the sum of the amounts obtained should be \$ 104; the apprentice should receive twice as much as the boy and the master workman five times as much as the boy. That is, we should have

$$13 + 26 + 65 = 104, \quad 26 = 2 \cdot 13 \quad \text{and} \quad 65 = 5 \cdot 13.$$

Rules for Solving Problems. From the preceding illustrations we see how much more simply problems may be solved by algebra than by arithmetic.

The following rules will help in solving problems:

(1) *Represent the unknown quantity by a letter.*

(2) *Translate the words of the problems into an equation involving this letter.*

(3) *Solve this equation by use of Principle VI.*

(4) *Verify the solution by showing that the requirements of the problem are fulfilled.*

ORAL PROBLEMS

1. Three times a certain number is 24. What is the number?

If x is the number, then $3x = 24$.

2. If $\frac{1}{3}$ of a certain number is 5, find the number. First state the equation.

In each of the following, state the equation and then give the required number.

3. If 6 is added to a certain number, the sum is 22. Find the number.

4. If 8 is subtracted from a certain number, the remainder is 10. Find the number.

5. Twice a certain number added to the number itself gives 12. Find the number.

6. Three times a number less the number itself equals 18. Find the number.

7. Twice a number plus three times the number equals 25. Find the number.

8. Six times a number less 4 times the number equals 8. Find the number.

9 If $3x$ plus $4x$ equals 42, find the value of x.

10. If $2a$ is subtracted from $9a$, the remainder is 21. Find the value of a.

11. If $7x$ is subtracted from $12x$, the remainder is 15. Find the value of x.

12. If $4y$ is added to $6y$, the sum is 100. Find the value of y.

13. If $3x$, $4x$, and $8x$ are added, the sum is 30. Find the value of x.

14. Twice a number plus three times the number plus four times the number is 36. Find the number.

15. If n is a number, how do you represent 10 times that number?

16. If n is a number, how do you represent that number plus 3 times itself?

17. If n is a number, how do you represent 5 times that number plus 3 times the number plus 8 times the number?

WRITTEN PROBLEMS

Solve the following problems by means of equations, and check each result by testing whether it satisfies the conditions of the problem.

1. Five times a certain number equals 80. What is the number?

2. Twelve times a number equals 132. What is the number?

3. A tank holds 750 gallons. How long will it take a pipe discharging 15 gallons per minute to fill the tank?

4. The cost of paving a block on a certain street was \$7 per front foot. How long was the block, if the total cost was \$4620?

5. A city lot sold for \$7500. What was the frontage, if the selling price was \$225 per front foot?

6. An encyclopedia contains 18,000 pages. How many volumes are there, if they average 750 pages to the volume?

7. For how many years must $3500 be invested at 6% simple interest to yield $2205?

8. At what rate must $2500 be invested for 3 years in order to yield $412.50?

Suggestion. By the conditions of the problem, $2500 \times r \times 3 = 412.50$.

9. At what rate must $6800 be invested for 7 years in order to yield $2380?

10. How many dollars must be invested for 5 years at $4\frac{1}{2}\%$ simple interest to yield $351?

Suggestion. By the conditions of the problem, $p \times .04\frac{1}{2} \times 5 = 351$.

11. How many dollars must be invested for 6 years at $4\frac{3}{4}\%$ simple interest to yield $2422.50?

12. A cut in an embankment is 500 yards long and 4 yards deep. How wide is it if 18,760 cubic yards are removed?

Suggestion. By the conditions of the problem, $500 \times w \times 4 = 18{,}760$.

13. How deep is a rectangular cistern which holds 500 cubic feet of water, if it is 6 feet wide and 8 feet long?

14. How long is a box containing 2240 cubic inches, if its width is 14 inches and its depth 10 inches?

15. The greater of two numbers is 5 times the less, and their sum is 180. What are the numbers?

16. A number increased by twice itself, 4 times itself, and 6 times itself, becomes 429. What is the number?

17. A father is 3 times as old as his son, and the sum of their ages is 48 years. How old is each?

18. In a company there are 39 persons. The number of children is twice the number of grown people. How many are there of each?

34. Further Hints for Translating Problems into Equations. Skill in solving problems depends upon attention to the following points:

(1) *Read and understand* clearly the statement of the problem, as it is given in words.

(2) *Represent the unknown number* by some letter, say the initial letter of a word, which will keep its meaning in mind. If there are more unknown numbers than one, try to express the others in terms of the letter first selected.

(3) Form two algebraic expressions which, according to the conditions of the problem, represent the same number, and set them equal to each other, *thus forming an equation.*

ORAL EXERCISES

1. If n is a number, represent in symbols a number 7 greater than n; 5 less than n; 8 times as great as n; one third as great as n.

2. Write in symbols n increased by k; n decreased by k; n multiplied by k; n divided by k.

3. If the sum of two numbers is 10 and one of them is x, what is the other number?

4. If two numbers differ by 6 and the smaller is x, what is the other number?

5. If two numbers differ by 6 and the greater is x, what is the other number?

6. If A has m dollars, and B has 15 dollars more than A, how do you represent B's money? If C's money is twice B's, how do you represent C's money?

7. A father is 34 years older than his son. If x represents the age of the son, how do you represent the father's age? How do you represent their ages 5 years ago?

8. If n is an integer, how do you represent the next higher integer? The second higher? The next lower? The second lower?

NOTE. — The whole numbers or integers are obtained by ordinary counting, beginning with *one*. If any integer such as 12 is given, the next higher integers may be represented by $12 + 1$, $12 + 2$, $12 + 3$, and so on. The next lower integers may be represented by $12 - 1$, $12 - 2$, $12 - 3$, and so on.

9. What is the value of $2n$, for $n = 1, 2, 3, 4, 5, 6$, etc.? If n is any integer, the number represented by $2n$ is called an *even* integer *since it contains the factor 2.*

10. If $2n$ is any even integer, represent the next higher even integer.

11. Represent each of four consecutive even integers, the smallest of which is $2n$.

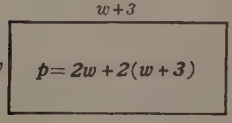
12. What is the value of $2n + 1$, for $n = 1, 2, 3, 4, 5$, etc.? If n is any integer, the number represented by $2n + 1$ is called an *odd* integer *since it does not contain the factor 2.*

13. If $2n + 1$ is any odd integer, represent the next higher odd integer.

14. Represent four consecutive odd integers, the smallest of which is $2n + 1$.

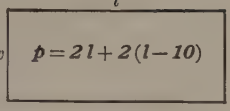
15. If x is a number, express in terms of x a number 5 less than 3 times x; also a number 5 times the remainder when 3 is subtracted from x.

16. If w and l are the width and length respectively of a rectangle, how do you represent its perimeter? (The perimeter of a rectangle means the sum of the lengths of its four sides.)



17. The length of a rectangle is 3 feet greater than its width. If w is the width, how do you represent its length? its perimeter?

18. Express the perimeter of a rectangle in terms of its length l if its width is 10 inches less than the length.



19. Express the perimeter of a rectangle in terms of its length l, if the length is 6 inches greater than the width.

WRITTEN PROBLEMS

Check each solution by finding whether the result satisfies the conditions stated in the problem:

1. Four times a certain number plus 3 times the number minus 5 times the number equals 48. What is the number?

2. One number is 4 times another, and their difference is 9. What are the numbers?

3. Find a number such that when 4 times the number is subtracted from 12 times the number, the remainder is 496.

4. Thirty-nine times a certain number, plus 19 times the number, minus 56 times the number, plus 22 times the number, equals 12. Find the number.

5. There are three numbers whose sum is 80. The second is 3 times the first, and the third twice the second. What are the numbers?

6. There are three numbers such that the second is 11 times the first and the third is 20 times the first. The difference between the second and third is 36. Find the numbers.

7. There are three numbers such that the second is 8 times the first and the third is 3 times the second. The third number less the second equals 48. Find the numbers.

8. The number of representatives and senators together in the United States Congress is 531. The number of representatives is 51 more than 4 times the number of senators. Find the number of each.

9. The area of Illinois is 6750 square miles more than 10 times that of Connecticut. The sum of their areas is 61,640 square miles. Find the area of each state.

10. Find three consecutive integers whose sum is 144.

11. Find four consecutive integers such that the last plus twice the first equals 48.

12. Find three consecutive even integers whose sum is 54.

13. Find three consecutive even integers such that 3 times the first is 12 greater than the third.

14. Find two consecutive integers such that 3 times the first plus 7 times the second equals 217.

15. Find two consecutive integers such that 7 times the first plus 4 times the second equals 664.

16. Find four consecutive odd integers such that 7 times the first equals 5 times the last.

17. Find the side of a square whose perimeter is 64 inches.

18. A rectangle is 4 inches longer than it is wide. Find its length and width if the perimeter is 40 inches.

19. A rectangle is twice as long as it is wide. Find its dimensions if the perimeter exceeds the length by 60.

20. The length of a rectangle is $1\frac{1}{2}$ times as great as its width. Find its dimensions if the perimeter exceeds the width by 40 inches.

21. The width of a rectangle is $\frac{1}{2}$ of its length and the perimeter exceeds the length by 50 inches. Find its dimensions.

22. The melting point of iron is 450 degrees centigrade higher than 5 times that of tin. Three times the number of degrees at which iron melts plus 7 times the number at which tin melts equals 6410. Find the melting point of each metal.

23. How many dollars will amount to $620 in 4 years at 6 % simple interest?

Solution. From arithmetic we have

$$amount = principal + interest,$$

or
$$a = p + i = p + prt.$$

Hence, by the conditions of the problem,

$$620 = p + .06 \times 4 \times p.$$

By Principle II. $$620 = p(1 + .24).$$

By *D*, $$p = \frac{620}{1.24} = 500.$$

24. Find what principal invested for 6 years at $4\frac{1}{2}$ % simple interest will amount to $1270.

25. Find what principal invested for 12 years at $5\frac{1}{2}$ % simple interest will amount to $4150.

26. How much must be invested at 6 % interest to amount to $2650 at the end of one year?

27. How much must be invested at 5 % simple interest to amount to $2025 at the end of 7 years?

28. At what rate of interest per year must $800 be invested to amount to $1000 in 5 years?

HISTORICAL NOTE

Representation of Unknown Numbers. The historical beginnings of algebra are found in the attempted solution of problems. The earliest work known on algebra is by an Egyptian priest, Ahmes, who used "heap" to represent the unknown. One of his problems reads, "heap, its seventh, its whole, it makes nineteen," which means: Solve the equation $\frac{x}{7} + x = 19$.

Diophantus, a Greek (300 A.D.), used the letter *s* to stand for the unknown. The Arabians used the word "thing" and the early European algebraists often used the Latin equivalent *res*. François Vieta used capital letters such as A. The fame of Vieta was such that other writers were led to follow him. Though many writers before Vieta had, now and then, used letters to represent the unknowns, we may regard the final establishment of this custom to date from his work.

René Descartes (see page 239) fixed the custom of using the last letters of the alphabet for the unknowns.

REVIEW QUESTIONS

1. Define equation; identity. State in detail how the equation and the identity differ. Give an example of each.

2. What value of x satisfies the equation $x+4=9$? What value of x will satisfy the equation obtained by adding 7 to each member of this equation? by adding 12? 24?

3. If 4 be added to the first member of the equation $x+4=9$, and 6 to the second member, what value of x will satisfy the equation thus obtained?

4. If the *same* number is added to each member of an equation, is the resulting equation equivalent to the first equation? Illustrate by an example.

5. If *different* numbers are added to the members of an equation, is the resulting equation equivalent to the first equation? Illustrate by an example.

6. If the *same* number is subtracted from each member of an equation, is the resulting equation equivalent to the first equation? Illustrate by an example.

7. If *different* numbers are subtracted from the members of an equation, is the resulting equation equivalent to the first equation? Illustrate by an example.

8. Ask and answer questions, similar to the two preceding, about multiplying and dividing both members of an equation by the *same or different* numbers. Illustrate each by examples.

9. *Each step in solving an equation consists in changing one equation into another equivalent equation whose solution is more apparent than that of the original equation.*

Hence what operations may be performed in solving an equation?

10. State Principle VI in full.

François Vieta (Viète) (1540–1603) was a French lawyer who wrote on mathematics for his own amusement. He printed and distributed his mathematical papers at his own expense. He used letters systematically to represent numbers, and from his time on this became the universal custom in algebra.

Such characters as $+$, $-$, though used as early as 1489 by Johann Widmann, did not come into general use before the time of Vieta. His keenness of mind was once shown by his discovery of the key to a Spanish cipher consisting of more than 500 characters. In their astonishment the Spaniards appealed to the Pope, accusing Vieta of using Black Art.

CHAPTER III

POSITIVE AND NEGATIVE NUMBERS

35. **A New Kind of Number.** Thus far the numbers used have been precisely the same as in arithmetic, though their representation by means of letters and some of the methods used in operating upon them are peculiar to algebra.

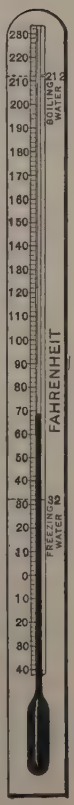
We now proceed to the study of a **new kind of number.**

Examples. What is the highest temperature you have ever seen recorded on the thermometer? the lowest?

In answering these questions you not only give certain numbers, but you attach to each a certain **quality.** The temperature is **above zero** or **below zero**; that is, the degrees on the thermometer are measured in **opposite** directions from a starting point which is marked zero.

Many other pairs of quantities possess such **opposite qualities**; for instance, motion to the *right* and to the *left*, *gain* and *loss*, *credit* and *debit*.

It has been found useful in mathematics to extend the number system of arithmetic so as to make it apply directly to cases like these. The opposite qualities involved are designated by the words **positive** and **negative.**



It is commonly agreed to call *above zero* positive and *below zero* negative; motion to the *right* positive and to the *left* negative; *credit* positive and *debit* negative; *gain* positive and *loss* negative.

36. **Positive and Negative Numbers.** The signs + and − stand respectively for the words *positive* and *negative*, and numbers marked with these signs are called *positive* and *negative numbers* respectively. See § 46.

Thus, 5° above zero is written +5°, and 15° below zero is written −15°.

When no sign of quality is written, the positive sign is understood.

E.g. +5° is usually written 5°.

Positive and negative numbers are sometimes called **signed numbers**, because each such number consists of a numerical part, together with a **sign of quality** expressed or understood.

The numerical part of a signed number is called its **absolute value.**

Thus, the absolute value of +3 and also of −3 is 3.

37. **Graphic Representation of Signed Numbers.** *The integers of arithmetic* may be arranged in a series beginning at zero and extending indefinitely toward the right.

Thus, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ···

The integers of algebra may be arranged in a series beginning at zero and extending indefinitely *both to the right and the left.*

Thus, ··· −5, −4, −3, −2, −1, 0, +1, +2, +3, +4, +5, ····

One of the most extensive uses of signed numbers is for marking the points on a straight line. On an unlimited straight line mark any point zero. On both sides of this point lay off equal divisions, as shown in the figure on page 45.

In order to describe the position of any one of these division points, we need not only an integer of arithmetic, to specify *how far* the given point is from the point marked zero, but also a **sign of quality** to indicate on *which side* of this point it is.

E.g. +6 marks the division point 6 units to the right of zero, and −5 marks the point 5 units to the left of zero. Such a diagram is called the **scale of signed numbers.**

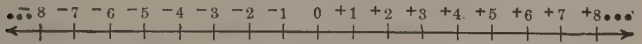
Fractions would of course be represented by points between the integral division points.

ADDITION OF SIGNED NUMBERS

38. Addition by Counting. In arithmetic two numbers may be added by starting with one number and counting *forward* as many units as there are units in the other number.

E.g. To add 3 to 5 we start with 5 and count 6, 7, 8.

Two signed numbers are added in the same manner except that the *direction*, forward or backward, in which we count, is determined by the sign $+$ or $-$, of the number which we are adding.



Thus, to add $+3$ to $+5$ begin at $+5$ and count 3 to the *right*.
To add -3 to $+5$ begin at $+5$ and count 3 to the *left*.
To add -3 to -5 begin at -5 and count 3 to the *left*.
To add $+3$ to -5 begin at -5 and count 3 to the *right*.
The results are as follows:

$+5 + +3 = +8$; $+5 + -3 = +2$; $-5 + -3 = -8$; $-5 + +3 = -2$.

$+5 + -3 = +2$ is read *positive* 5 *plus negative* 3 *equals positive* 2.
$-5 + -3 = -8$ is read *negative* 5 *plus negative* 3 *equals negative* 8.
In like manner, read the other two.

ORAL EXERCISES

Using the number scale above perform the following additions by counting.

1. $+3 + -1$.	11. $-2 + -3$.	21. $-4 + -3$.
2. $+4 + -2$.	12. $-3 + -5$.	22. $-3 + +7$.
3. $+5 + -3$.	13. $-4 + +6$.	23. $+8 + -9$.
4. $+5 + -5$.	14. $-5 + +5$.	24. $+5 + -3$.
5. $+5 + -7$.	15. $-7 + +8$.	25. $+7 + -5$.
6. $+5 + -8$.	16. $-6 + +5$.	26. $+8 + -7$.
7. $+8 + -7$.	17. $-6 + -2$.	27. $-3 + -4$.
8. $+6 + -4$.	18. $-7 + +7$.	28. $-1 + +7$.
9. $+7 + -7$.	19. $-8 + +4$.	29. $-2 + -5$.
10. $+9 + -5$.	20. $-8 + +6$.	30. $-7 + +5$.

39. Further Illustrations of Signed Numbers. The meaning of positive and negative numbers is further explained in the following

Illustrative Problems: 1. If a man gains \$ 1500 and then loses \$ 800, what is the net result? *Answer,* \$ 700 gain.

In this case the result is obtained by subtracting 800 from 1500. Yet this is not really a problem in subtraction but in addition. That is, we are not asking for the difference between \$ 1500 gain and \$ 800 loss, but for the *net result* when the gain and the loss are taken together, or the sum of the profit and the loss. Hence, we say \$ 1500 gain + \$ 800 loss = \$ 700 gain, or using positive and negative signs,

$$+1500 + {}^-800 = {}^+700.$$

2. The assets of a commercial house are \$ 250,000, and the liabilities are \$ 275,000. What is the net financial status of the house? *Answer,* \$ 25,000 net liabilities.

Thus,
\$ 250,000 assets + \$ 275,000 liabilities = \$ 25,000 net liabilities.
Or $+250{,}000 + {}^-275{,}000 = {}^-25{,}000.$

3. The thermometer rises 18 degrees and then falls 28 degrees. What direct change in temperature would produce the same result? *Answer,* 10 degrees fall.

Thus, 18° rise + 28° fall = 10° fall.
Or $+18 + {}^-28 = {}^-10.$

4. A man travels 700 miles east and then 400 miles west. What direct journey would bring him to the same final destination? *Answer,* 300 miles east.

Thus, 700 miles east + 400 miles west = 300 miles east.
Or $+700 + {}^-400 = {}^+300.$

Positive and negative numbers may be written in columns and added. Thus the above examples would stand as follows:

+1500	+250,000	+18	+700
−800	−275,000	−28	−400
+700	−25,000	−10	+300

The preceding exercises illustrate

Principle VII

40. **Rule.** *To add two numbers with like signs, find the sum of their absolute values, and prefix to this their common sign.*

To add two numbers with opposite signs, find the difference of their absolute values, and prefix to this the sign of that one whose absolute value is the greater.

In case the signs of two numbers are opposite and their absolute values are equal, their sum is zero.

41. **Signed Numbers which Cancel each other.** We have seen that in finding the sum of two numbers whose signs are *opposite*, they tend to cancel each other.

Thus, in adding +5 and −8, the +5 cancels −5 out of −8 and leaves −3 as the sum. In adding −5 and +8, the −5 cancels +5 and leaves +3 as the sum.

ORAL EXERCISES

In this manner find the sums of the following:

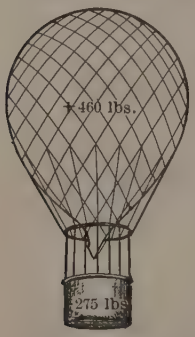
1. +7, −4	7. −8, −3	13. +7, −19	19. −25, +35	25. −9, −12
2. −9, +7	8. −9, −4	14. −14, +8	20. −15, +24	26. −12, −8
3. −8, −3	9. −4, +8	15. +30, −40	21. −12, −18	27. +12, −16
4. +12, −7	10. +9, −15	16. +20, −50	22. +20, −50	28. +18, −12
5. −8, +12	11. +13, −17	17. +15, −25	23. +14, −21	29. +20, −35
6. +12, 7	12. +12, −14	18. +10, −20	24. +18, −24	30. +35, −20

Algebraic Sum. The sum of signed numbers obtained as in the preceding exercises is called their *algebraic sum.*

Signed numbers find application in any situation where *opposite qualities* of the kind here considered are present. Besides those already mentioned, other instances occur in the applications below.

WRITTEN EXERCISES

1. A balloon which exerts an upward pull of 460 pounds is attached to a car weighing 275 pounds. What is the net upward or downward pull? Express this as a problem in addition, using positive and negative numbers.



Solution. 460 lbs. upward pull plus 275 lbs. downward pull equals 185 lbs. net upward pull. Using positive numbers to represent upward pull and negative numbers to represent downward pull, this equation becomes

$$+460 + -275 = +185.$$

In each of the following translate the solution into the language of algebra by means of signed numbers as in Example 1.

2. A balloon which exerts an upward pull of 600 pounds has a 450-pound weight attached to it. What is the net upward or downward pull?

3. A man's property amounts to $45,000 and his debts to $52,000. What is his net debt or property?

4. The assets of a bankrupt firm amount to $245,000 and the liabilities to $325,000. What are the net assets or liabilities?

5. A man can row a boat at the rate of 6 miles per hour. How fast can he proceed against a stream flowing at the rate of $2\frac{1}{2}$ miles per hour? against one flowing 7 miles per hour?

6. A steamer which can make 12 miles per hour in still water is running against a current flowing 15 miles per hour. How fast and in what direction does the steamer move?

42. Averaging Signed Numbers. Half the sum of two numbers is called their *average*. Thus 6 is the average of 4 and 8. Similarly, the *average of three numbers* is one third of their sum, and in general *the average of n numbers* is the sum of the numbers divided by *n*.

Find the average of each of the following sets:

1. 10, 12, 14, 16, 18.
2. 7, 9, 11, 13, 15, 8.

The average gain or loss per year for a given number of years is the *algebraic sum* of the yearly gains and losses divided by the number of years.

Illustrative Problem. A man lost \$ 400 the first year, gained \$ 300 the second, and gained \$ 1000 the third. What was the average loss or gain?

Solution. $(-400 + +300 + +1000) \div 3 = +900 \div 3 = +300$.
That is, the average gain is \$ 300.

WRITTEN PROBLEMS

1. Find the average of \$ 1800 loss, \$ 3100 loss, \$ 6800 gain, \$ 10,800 loss, and \$ 31,700 gain.

Suggestion. Add all the positive members separately and all negative members separately. Then combine the two sums.

2. Find the average of \$ 180 gain, \$ 360 loss, \$ 480 loss, \$ 100 gain, \$ 700 gain, \$ 400 gain, \$ 1300 loss, \$ 300 gain, \$ 4840 gain, and \$ 12,000 gain.

Find the average yearly temperatures at the following places, the monthly averages having been recorded as given below:

3. For New York City: $+29°$, $+33°$, $+39°$, $+46°$, $+53°$, $+63°$, $+67°$, $+67°$, $+61°$, $+52°$, $+47°$, $+41°$.

4. For St. Vincent, Minnesota: $-5°$, $0°$, $+15°$, $+35°$, $+55°$, $+60°$, $+66°$, $+63°$, $+55°$, $+40°$, $+22°$, $+5°$.

5. For Nerchinsk, Siberia: $-23°$, $-13°$, $-10°$, $+35°$, $+55°$, $+70°$, $+70°$, $+64°$, $+50°$, $+30°$, $+5°$, $-15°$.

SUBTRACTION OF SIGNED NUMBERS

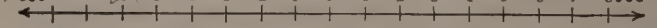
43. Subtraction is the process of finding the number which added to the subtrahend will equal the minuend.

Thus, to subtract 3 from 8 is to find the number which added to 3 will make 8. That is, $8 - 3 = 5$ because $3 + 5 = 8$.

Then the test for subtraction is

Remainder + Subtrahend = Minuend.

In subtracting signed numbers we may start on the scale at the point indicated by the subtrahend and find *how far* and in *which direction* we must count in order to reach the minuend.



Thus $+8 - +5 = +3$, because from $+5$ to $+8$ is 3 in the *positive* direction.
$+8 - -5 = +13$, because from -5 to $+8$ is 13 in the *positive* direction.
$-8 - -5 = -3$, because from -5 to -8 is 3 in the *negative* direction.
$-8 - +5 = -13$, because from $+5$ to -8 is 13 in the *negative* direction.

Similarly, perform the following subtractions:

1. $-4 - -2$.
2. $-5 - +6$.
3. $-8 - -4$.
4. $7 - -6$.
5. $8 - -8$.
6. $-4 - -7$.
7. $-7 - -4$.
8. $-3 - -8$.

44. A Short Rule for Subtraction. Since $+8 - -5 = +13$, and since $+8 + +5 = +13$, it follows that *subtracting* -5 from $+8$ gives the same result as *adding* $+5$ to $+8$. Similarly, $-8 - -5 = -3$ and $-8 + +5 = -3$.

Hence, *subtracting a negative* number is equivalent to *adding a positive* number with the same absolute value.

Since $+8 - +5 = +3$, and since $+8 + -5 = +3$, it follows that subtracting $+5$ from $+8$ gives the same result as *adding* -5 to $+8$ Similarly, $-8 - +5 = -13$ and $-8 + -5 = -13$.

Hence, *subtracting a positive* number is equivalent to *adding a negative* number with the same absolute value.

These statements are illustrated by such facts as: *Removing a debt* is equivalent to *adding property* and *removing property* is equivalent to *adding debt.*

Positive and negative numbers may be written in columns and subtracted.

Thus	+8	+8	−8	−8
	+5	−5	−5	+5
	+3	+13	−3	−13

ORAL EXERCISES

Perform the following subtractions by changing the sign of the subtrahend and adding:

1. −5	3. +3	5. −16	7. −16	9. +16
−2	−5	−12	+4	−12
2. −4	4. +57	6. +12	8. −19	10. −48
+1	−32	+50	−24	−23

The preceding exercises illustrate

Principle VIII

45. Rule. *To subtract one signed number from another signed number, change the sign of the subtrahend and then add it to the minuend.*

The change in the sign of the subtrahend may be made *mentally* without rewriting the problem. Check by showing that *Remainder* + *Subtrahend* = *Minuend.*

WRITTEN EXERCISES

Perform the following subtractions, changing the signs of the subtrahends *mentally:*

1. −10 − −5.	7. +6 − −14.	13. −78 − −37.
2. −15 − +5.	8. +7 − −9.	14. +57 − +84.
3. +20 − −15.	9. −11 − +6.	15. −48 − −31.
4. +11 − +3.	10. −21 − −6.	16. −39 − −95.
5. −11 − +5.	11. +93 − +22.	17. −91 − −3.
6. −17 − −20.	12. +17 − −13.	18. −38 − +74.

Subtraction always Possible in Algebra. In arithmetic subtraction is possible only when the subtrahend is less than, or equal to, the minuend.

E.g. In arithmetic we cannot subtract 5 from 2 since there is no positive number which added to 5 gives 2.

However, in algebra, by means of negative numbers we can as easily perform the subtraction, 2 minus 5, as 5 minus 2.

Thus, $2 - 5 = {}^{-}3$, since ${}^{-}3 + 5 = 2$.

Similarly, $0 - {}^{+}5 = {}^{-}5$, since ${}^{-}5 + {}^{+}5 = 0$, and ${}^{-}1 - {}^{+}6 = {}^{-}7$ since ${}^{-}7 + {}^{+}6 = {}^{-}1$.

46. **Double Use of the Signs + and −.** In § 36 we agreed that when no sign of quality is written, the sign $^{+}$ is understood. Hence we may write:

$$^{+}8 + {}^{+}5 = 8 + 5. \tag{1}$$

$$^{+}8 - {}^{+}5 = 8 - 5. \tag{2}$$

By Principle VII, we have

$$^{+}8 + {}^{-}5 = {}^{+}8 - {}^{+}5 = 8 - 5. \tag{3}$$

By Principle VIII, we have

$$^{+}8 - {}^{-}5 = {}^{+}8 + {}^{+}5 = 8 + 5. \tag{4}$$

These examples show how we may dispense with the special signs of quality $^{+}$, or $^{-}$, as follows:

1. *Positive numbers are written without any sign indicating quality except where special emphasis is desired, in which case the sign + is used.*

2. *A negative number when standing alone is preceded by the sign −. Thus* $^{-}5$ *is written* -5.

3. *When a negative number is combined with other numbers, its quality is indicated by the sign − with parentheses inclosing it.*

Thus, $8 + {}^{-}5$ is written $8 + (-5)$,
and $8 - {}^{-}5$ is written $8 - (-5)$.

But in such cases it is customary to apply Principles VII and VIII and write at once $8 - 5$ instead of $8 + (-5)$ and $8 + 5$ instead of $8 - (-5)$.

WRITTEN EXERCISES

Work Examples 1–16 twice, first adding, then subtracting.

1. -25 / 14	5. -28 / -10	9. 13 / -20	13. 16 / 30
2. 25 / -14	6. 28 / -10	10. -13 / -20	14. -16 / 30
3. 25 / 14	7. -28 / 10	11. -13 / 20	15. -16 / -30
4. -25 / -14	8. 28 / 10	12. 13 / 20	16. 16 / -30

Perform the operations indicated.

17. $20-(-5)$	22. $15a-(-20a)$
18. $20+(-5)$	23. $16n-25n$
19. $20-(-30)$	24. $40s-(-30s)$
20. $20x-30x$	25. $15b-35b$
21. $15a-20a$	26. $43x-71x$

EXPLANATORY NOTE

On the Double Use of Signs. It is clear from the examples in this chapter that signed numbers are needed to represent actual conditions in life, as in case of the thermometer. While from now on such numbers will be distinguished, in accordance with universal custom, by the signs $+$, $-$, it should be understood that each of these signs is thus made to represent either one of two entirely different things, namely, an *operation* or a *quality*.

After we acquire some understanding of the matter, this double use of the signs seldom leads to confusion, since we can always tell from the context which use is meant. For example, in $5-3$, the sign $-$ means *subtraction*, while in $x=-3$ it means *negative*.

But for the sake of avoiding confusion at the outset, and to make clear that a negative number is not necessarily a *subtrahend* and that a positive number is not necessarily an *addend*, we have up to this time used the special signs $^+$, $^-$, which could be readily distinguished from the signs of addition and subtraction. They will now be discontinued.

WRITTEN EXERCISES

Perform the following indicated operations:

1. Find the value of $a + b$ if (1) $a = 4, b = -5$; (2) $a = -2, b = -7$; (3) $a = -6, b = 8$; (4) $a = 6, b = -10$.

2. Find the value of $a - b$ if (1) $a = 8, b = 8$; (2) $a = -3, b = -7$; (3) $a = 4, b = -9$; (4) $a = -3, b = 6$.

3. Find the value of $a + b + c$ if (1) $a = 3, b = -4, c = -1$; (2) $a = -7, b = 3, c = 2$; (3) $a = -1, b = -8, c = 10$.

4. Find the value of $a + b - c$ if (1) $a = 6, b = -4, c = 5$; (2) $a = -2, b = -4, c = -6$; (3) $a = 7, b = -8, c = -6$.

5. Find the value of $a - b - c$ if (1) $a = 3, b = 6, c = -2$; (2) $a = -6, b = 7, c = -12$; (3) $a = 8, b = 4, c = 8$.

6. Find the value of $-a - b + c$ if (1) $a = 7, b = -6, c = 4$; (2) $a = -2, b = -6, c = 2$.

7. Find the value of $-a + b - c$ if (1) $a = -1, b = -2, c = -3$; (2) $a = -8, b = 10, c = -2$.

8. Find the value of $-a - b - c$ if (1) $a = -3, b = -2, c = -1$; (2) $a = 6, b = -3, c = -9$.

Solve the following equations:

9. $x + 8 = 4$.

Suggestion. Subtract 8 from each member, or transpose 8 and change its sign.

10. $x + 3 = 7$.

11. $x - 9 = 1$.

12. $3 + x = 0$.

13. $x + 13 = 7$.

14. $-4 + x = -9$.

15. $-5 + x = 4$.

16. $-5 + x = 12$.

17. $-9 + x = -18$.

18. $-35 + x = 17$.

19. $17 + x = -35$.

20. $x - 14 = -18$.

21. $x - 25 = 16$.

22. $x + 4 = 1$.

23. $x - 7 = -15$.

24. $-21 + x = -17$.

25. $-16 + x = -18$.

26. $-12 + x = -20$.

ORAL EXERCISES

Add the following:

1.	18	4.	− 9	7.	81	10.	16	13.	− 9
	− 7		18		− 72		− 24		− 7
	+ 11		9		− 9		− 8		− 16
2.	− 6	5.	− 24	8.	46	11.	− 16	14.	− 8
	5		17		− 38		− 7		14
	− 1		− 9		8		− 28		6
3.	− 7	6.	− 18	9.	− 36	12.	− 8	15.	− 17
	− 14		42		24		− 9		4
	− 21		24		− 12		− 1		− 13

Subtract the following:

16.	30	19.	4	22.	7	25.	− 10	28.	10
	− 15		− 12		− 7		20		− 40
	45		16		+ 14		− 30		+ 50
17.	− 4	20.	6	23.	− 20	26.	− 10	29.	70
	8		− 8		14		− 20		− 40
	− 12		+ 12		+ 6		10		110
18.	− 4	21.	− 7	24.	10	27.	− 15	30.	− 16
	− 12		7		− 20		15		8
	− 8		− 14		− 30		30		− 24

Solve each of the following, using an equation involving positive and negative numbers.

31. A dove which can fly 40 miles per hour in calm weather is flying against a hurricane blowing at the rate of 60 miles per hour. How fast and in what direction is the dove moving?

32. If of two partners, one loses $ 1400 and the other gains $ 3700, what is the net result to the firm?

33. A man's income is $ 2400 and his expenses $ 1500 per year. What is the net result for the year?

34. A man loses $ 800 and then loses $ 600 more. What is the combined loss? Indicate the result as the sum of two negative numbers.

MULTIPLICATION OF SIGNED NUMBERS

47. The multiplication of signed numbers is illustrated by the following problems.

Illustrative Problem. A balloonist, just before starting, makes the following preparations: (*a*) He adds 9000 cubic feet of gas with a lifting power of 75 pounds per thousand cubic feet. (*b*) He takes on 8 bags of sand, each weighing 15 pounds. How do these operations affect the buoyancy of the balloon?

Solution. (*a*) A lifting power of 75 pounds is indicated by $+75$, and adding such a power 9 times is indicated by $+9$. Hence, $+9(+75) = +675$, or 675 is the total *lifting* power added.

(*b*) A weight of 15 pounds is indicated by -15, and adding 8 such weights is indicated by $+8$. Since the total weight added is 120 pounds, we have $+8(-15) = -120$, which is the total *depressing* power added.

Illustrative Problem. During the course of his journey the balloonist opens the valve and allows 2000 cubic feet of gas to escape, and later throws overboard 4 bags of sand. How does each of these operations affect the buoyancy of the balloon?

Solution. (*a*) The gas, being a lifting power, is positive, but the removal of 2000 cubic feet of it is indicated by -2, and the result is a depression of the balloon by 150 pounds; that is, $-2 \cdot (+75) = -150$.

(*b*) The removal of 4 weights is indicated by -4, but the weights themselves have the negative quality of downward pull. Hence to remove 4 weights of 15 pounds each is equivalent to increasing the buoyancy of the balloon by 60 pounds; that is, $-4 \cdot (-15) = +60 = 60$.

Notion of Multiplication Extended. These examples illustrate a natural extension of multiplication in arithmetic.

E.g. Just as $3 \cdot 4 = 4 + 4 + 4 = 12$, so $3 \cdot (-4) = -4 + (-4) + (-4) = -12$. Hence we write $+3 \cdot (+4) = +12 = 12$, and $+3 \cdot -4 = -12$.

Again, just as we take the multiplicand *additively* when the multiplier is a positive integer, so we take it *subtractively* when the multiplier is a negative integer.

E.g. $-3 \cdot (+4) = -(+4) - (+4) - (+4) = -12$,
and $-3 \cdot (-4) = -(-4) - (-4) - (-4) = -(-12) = +12$.
Hence we write $-3 \cdot (+4) = -12$ and $-3 \cdot (-4) = +12 = 12$.

Two numbers may be conveniently placed in columns for multiplying.

Thus,

$+4$	$+4$	-4	-4
$+3$	-3	$+3$	-3
$+12$	-12	-12	$+12$

ORAL EXERCISES

Explain the following indicated multiplications and find the product in each case.

1. $-3\cdot(-10)$. $+30$
2. $10\cdot(-3)$. -30
3. $10\cdot(-5)$. -50
4. $-7\cdot(-8)$. 56
5. $12\cdot(-4)$. -48
6. $-7\cdot(-6)$. 42
7. $-5\cdot(-12)$.
8. $-5\cdot(-8)$.
9. $-8\cdot 6$.

The preceding exercises illustrate

Principle IX

48. **Rule.** *If two numbers have the same sign, their product is positive; if they have opposite signs, their product is negative.*

In applying this principle observe that the sign of the product is obtained quite independently of the absolute value of the factors.

E.g. $\frac{3}{4}\cdot(-5)=-(\frac{15}{4})=-3\frac{3}{4}$; $-12\cdot(-3\frac{1}{2})=+42=42$.

ORAL EXERCISES

Multiply the following:

1.	7	5.	6	9.	9
	-4 -28		-8 -48		-10 -90
2.	-5	6.	9	10.	8
	6 -30		-8 -72		-6 $+48$
3.	8	7.	10	11.	-5
	-5 -40		-8 -80		4 -20
4.	-12	8.	20	12.	-6
	4 -48		6 120		5 -30

WRITTEN EXERCISES

Multiply the following pairs of numbers:

1.	-14 -3	6.	45 6	11.	-5.4 -4
2.	-14 3	7.	25 4	12.	$15a$ -4
3.	14 -3	8.	-25 4	13.	$-22x$ -3
4.	-40 5	9.	-25 -4	14.	$-18m$ 4
5.	-35 -5	10.	-4.2 6	15.	$-16ab$ -2

49. **The product of several signed numbers** is found as illustrated in the following:

$-2 \cdot 5 \cdot (-3) \cdot (-4) \cdot 6 = -10 \cdot (-3) \cdot (-4) \cdot 6 = 30 \cdot (-4) \cdot 6 = -120 \cdot 6 = -720$. That is, the first two factors are multiplied together, then their product is multiplied by the next factor, and so on, until all the factors are multiplied.

Since the product of all positive factors is positive, the final sign depends upon the number of negative factors. If this number is *even*, the product is positive; if it is *odd*, the product is negative.

E.g. If there are 3 negative factors, the product is negative; if there are 4, it is positive.

In the following exercises determine the sign of the product before finding its absolute value.

ORAL EXERCISES

1. $-4 \cdot 3 \cdot (-2) \cdot (-2)$.
2. $-2 \cdot (-3) \cdot (-5) \cdot 3$.
3. $-5 \cdot (-3)(-2)(-4)(-1)$.
4. $-5 \cdot (-4) \cdot 3 \cdot (-2)$.
5. $8 \cdot (-9) \cdot (-1) \cdot (-2)$.
6. $-5 \cdot (-2) \cdot (-3) \cdot (-4)$.

DIVISION OF SIGNED NUMBERS

50. **Test for Division.** In arithmetic we test the correctness of division by showing that the quotient multiplied by the divisor equals the dividend.

E.g. $27 \div 9 = 3$, because $9 \cdot 3 = 27$.

Hence **division** may be defined as the process of finding one of two factors when their product and the other factor are given.

The given product is the **dividend**, the given factor is the **divisor**, and the factor to be found is the **quotient.**

In dividing signed numbers the above test determines the *sign* of the quotient as well as its *absolute value.*

E.g. $-42 \div (+6) = -7$, because $-7 \cdot (+6) = -42$;
also $-42 \div (-6) = +7$, because $+7 \cdot (-6) = -42$.

So in every case the test of the correctness of division is:

Quotient $\times$ *Divisor* = *Dividend.*

Find the following quotients and check as above:

1. $-25 \div 5$.
2. $-ab \div a$.
3. $5xy \div (-x)$.
4. $-9rs \div (-3)$.
5. $75y \div (-15)$.
6. $-121x \div 11$.

The preceding exercises illustrate

Principle X

51. **Rule.** *The quotient of two signed numbers is positive if the dividend and divisor have like signs, negative if they have opposite signs.*

ORAL EXERCISES

1. $\frac{-6}{-3}$	4. $\frac{-6}{-3}$	7. $\frac{-8}{4}$	10. $\frac{20}{-10}$	13. $\frac{-48}{16}$
2. $\frac{6}{3}$	5. $\frac{8}{-4}$	8. $\frac{-15}{-5}$	11. $\frac{-20}{10}$	14. $\frac{-48}{-16}$
3. $\frac{-6}{3}$	6. $\frac{-8}{-4}$	9. $\frac{-15}{5}$	12. $\frac{-20}{-10}$	15. $\frac{-48}{16}$

EXERCISES

Perform the following indicated divisions, and check the first ten by multiplying quotient by divisor. Do Examples 1–21 orally.

1. $\frac{-28}{7}$.
2. $\frac{-42}{-6}$.
3. $\frac{51}{-17}$.
4. $\frac{-21}{-3}$.
5. $\frac{-75}{5}$.
6. $\frac{-16}{-1}$.
7. $\frac{-49}{7}$.
8. $\frac{64}{-4}$.
9. $\frac{4\cdot(-9)}{-3}$.
10. $\frac{-3\cdot 8}{-4}$.
11. $\frac{10\cdot(-9)}{-5}$.
12. $\frac{3\cdot(-4)\cdot(-2)\cdot 6}{-3}$.
13. $\frac{-42a}{-3}$.
14. $\frac{-42x}{3}$.
15. $\frac{42x}{-3}$.
16. $\frac{75abc}{-a}$.
17. $\frac{-100xy}{-x}$.
18. $\frac{-320y}{80}$.
19. $\frac{25xy}{-x}$.
20. $\frac{-196mn}{-14}$.
21. $\frac{-39ab}{-13a}$.

22. A man lost \$300, \$500, and \$700 during three consecutive months. Express his average monthly loss as a quotient.

23. During five consecutive days the minimum temperature was $-5°$, $-8°$, $-10°$, $-4°$, $-6°$ respectively. Find the average of these temperatures.

24. A trader lost \$250 in each of three months and gained \$75 during each of the four succeeding months. Find the average gain or loss for the seven months.

25. On a cold day the following temperatures was observed: $-20°$, $-16°$, $-12°$, $-4°$, $2°$, $8°$, $4°$, $-2°$. Find the average of these readings.

26. Find the average of the numbers: 280, -960, -840, -360, 860, -260, -180, 530, and -480. See suggestions under Example 1, page 49.

52. The Principles applied to Signed Numbers. While Principles I–V were studied in connection with unsigned, or arithmetic numbers only, it is now important to note that they all apply to *signed* numbers as well.

In the statement of these principles the word *number* will from now on be understood to refer either to the ordinary numbers of arithmetic or to the signed numbers of algebra, as occasion may require. It should also be noticed that the numbers of arithmetic are used as freely in algebra as in arithmetic. It is only when we wish to distinguish them from negative numbers that they are called *positive* numbers.

ORAL EXERCISES

If $a = 6$, $b = 4$, $c = -2$, evaluate the following:

1. $a(b + c)$.
2. $a(b - c)$.
3. $a(c - b)$.
4. $c(a + b)$.
5. $c(b - a)$.
6. $c(a - b)$.

WRITTEN EXERCISES

1. Find the quotient $\dfrac{2(-3)(-4)(-6)(-8)(-2)}{4(-12)(-16)}$.

Solution. There are five negative factors in the dividend and two in the divisor. Hence the sign of the dividend is $-$ and that of the divisor is $+$. (See § 49.) Therefore the sign of the quotient is $-$. See § 51. We now cancel as if all the factors were positive and prefix the negative sign to the quotient, obtaining -3.

In like manner find the following quotients:

2. $\dfrac{a(-b)(-e)(-d)}{ab(-e)(-d)}$.
3. $\dfrac{3(-12)(-18)(-32)}{6 \cdot 8(-24)(-3)}$.
4. $\dfrac{-8 \cdot 9(-3)(-4)}{-3(-4)(-2)(-6)}$.
5. $\dfrac{ab(-c)(d)(-e)}{(-a)(-b)(-d)(e)}$.
6. $\dfrac{x(-y)(-z)(-4)}{xy(-4z)}$.
7. $\dfrac{2x(-y)(-z)(-v)}{-xv(-z)y(-3)}$.

The **number system of algebra**, so far as we have now studied it, consists of the *numbers of arithmetic* together with the *negative numbers.*

HISTORICAL NOTE

The Development of Negative Numbers. The Greeks had no conception of a negative number as distinct from a number to be subtracted. Diophantus states that in the multiplication of $(a - b)$ by $(c - d)$, *a subtraction multiplied by a subtraction gives an addition.* That is, $(-b)(-d) = bd$. But in applying the rule he takes care that a is greater than b, and c greater than d. However, the Hindus appear to have had quite clear notions of a purely "negative number" as distinct from a number to be subtracted. They recognized the difference between positive and negative numbers by attaching to one the idea of debt and to the other that of assets, or by letting them represent distances in opposite directions. The Arabs, however, failed to understand the negative numbers and did not include them in the algebra which they brought to Europe. (See page 32.) Until the beginning of the seventeenth century, mathematicians dealt almost exclusively with positive numbers.

Thomas Harriot, an Englishman (1560–1621), was the first to write a negative number all by itself. Thus an equation like $x = -3$ was never written by any of Harriot's predecessors.

The negative numbers were brought permanently into mathematics by René Descartes. (See page 239.) Trying to number all the points of a complete straight line, Descartes was compelled to start at some point and **number in both directions**. Then it became convenient to distinguish the numbers on the two sides of this starting point as positive and negative, respectively.

Sir Isaac Newton (see page 101) was the first to let a letter stand for any number, negative as well as positive. In such a formula as $a(b + c) = ab + ac$, the predecessors of Newton had restricted the letters to represent any **positive** numbers, while Newton regarded the letters as representing *any numbers whatever,* either positive or negative. This was of very great importance, since it greatly reduced the number of formulas required.

Negative numbers appeared "absurd" or "fictitious" until a **visual or graphical representation of them** was discovered. Cajori in his history of elementary mathematics says: "Omit all illustrations by lines, thermometers, etc., and negative numbers will be as obscure to modern students as they were to the early algebraists." From the experience of the early mathematicians it would appear that if the pupil wishes really to understand positive and negative numbers, he must study with care applications such as are given in the first part of this chapter.

INTERPRETATION AND USE OF NEGATIVE NUMBERS

53. A negative result obtained in solving a problem may have a natural interpretation, or it may indicate that the conditions of the problem are impossible.

A similar statement holds regarding fractional or zero answers in arithmetic. For example, if we say there are twice as many girls as boys in a schoolroom and 35 pupils in all, the number of boys would be $35 \div 3 = 11\frac{2}{3}$, which indicates that the conditions of the problem are impossible.

Again, if three buildings cost in all \$18,500, and if the second cost \$9000 more than the first, and the third \$9500 more than the second, then the cost of the first building would be zero, which is impossible.

Illustrative Problem. The crews on three steamers together number 94 men. The second has 40 more than the first, and the third 20 more than the second. How many men in each crew?

Solution. Let n = number of men in first crew.

Then, $n + 40$ = number of men in second crew,

and $n + 40 + 20$ = number of men in third crew.

Hence, $n + n + 40 + n + 40 + 20 = 94$,

and $3n + 100 = 94$.

$3n = -6$.

$n = -2$.

Here the negative result indicates that the conditions of the problem are *impossible.*

Illustrative Problem. A real estate agent gained \$8400 on four transactions. On the first he gained \$6400, on the second he lost \$2100, on the third he gained \$5000. Did he lose or gain on the fourth transaction, and how much?

Solution. Since we do not know whether he gained or lost on the fourth transaction, we represent the unknown number by n, which may be positive or negative, as will be determined by the solution of the problem.

Then we have $6400 + (-2100) + 5000 + n = 8400$. (1)

Hence, by VII, *F*, $9300 + n = 8400$. (2)

By S, $n = 8400 - 9300$. (3)

By VIII, $n = -900$. (4)

In this case the negative result indicates that the conditions of the problem are possible, and that there was a *loss* on the fourth transaction.

PROBLEMS

In the following problems give the solutions in full and state all principles used.

In case a negative answer is found, state whether this answer has a natural interpretation, or whether it indicates that the conditions of the problem are impossible.

1. A man gains $ 2100 during one year. During the first three months he loses $ 125 per month, then gains $ 500 per month during the next five months. What is the average gain or loss per month during the remaining four months?

2. A man rowing against a swift current goes 9 miles in 5 hours. The second hour he goes two miles less than the first, the third three miles more than the second, the fourth one mile more than the third, and the fifth one mile more than the fourth. How many miles did he go during each of the five hours?

3. There are three trees the sum of whose heights is 108 feet. The second is 40 feet taller than the first, and the third is 30 feet taller than the second. How tall is each tree?

In the next three examples find the average yearly temperatures, the average monthly temperatures being as here given:

4. Port Conger, off the northwest coast of Greenland: $-37°$, $-43°$, $-32°$, $-15°$, $14°$, $18°$, $35°$, $34°$, $25°$, $4°$, $-17°$, $-30°$.

5. Franz Joseph's Land: $-20°$, $-20°$, $-10°$, $0°$, $15°$, $30°$, $35°$, $30°$, $20°$, $10°$, $0°$, $-10°$.

6. North Central Siberia: $-60°$, $-50°$, $-30°$, $0°$, $15°$, $40°$, $40°$, $35°$, $30°$, $0°$, $-30°$, $-50°$.

7. A merchant gained an average of $ 2800 per year for 5 years. The first year he gained $ 3000, the second $ 1500, the third $ 4000, and the fourth $ 2400. Did he gain or lose during the fifth year, and how much?

8. A certain business shows an average gain of $4000 per year for 6 years. During the first five years the results were: $8000 loss, $10,000 gain, $7000 gain, $3000 gain, and $12,000 gain. Find the loss or gain during the sixth year.

9. A commercial house averaged $10,000 gain for 6 years. What was the loss or gain the first year if the remaining years show: $8000 gain, $24,000 gain, $2000 loss, $20,000 gain, and $30,000 gain, respectively?

REVIEW QUESTIONS

1. Name several pairs of opposite qualities all of which are conveniently described by the words *positive* and *negative*. What symbols are used to replace these words when applied to numbers?

2. When loss is added to profit, is the profit increased or decreased? What algebraic symbols may be used to distinguish the numbers representing profit and loss?

3. On the number scale indicate what is meant by $+2$; by -2. Indicate what is meant by the sign $+$ in $5+2$; by the sign $-$ in $5-2$; by the sign $-$ in $x=-2$.

4. Why do we call positive and negative numbers *signed* numbers? What is meant by the *absolute value* of a number?

5. State Principle VII in full.

6. How is the correctness of subtraction tested in arithmetic? Is the same test applicable to subtraction in algebra?

7. Illustrate the subtraction of positive and negative numbers by an example involving profit and loss.

8. Show by counting on the number scale that the result of subtraction gives the *distance* from subtrahend to minuend and that the sign of the remainder shows the *direction* from subtrahend toward the minuend. For example, use $8-(-5)$ and $-8-(+5)$ to illustrate this.

9. How do negative numbers make subtraction possible in cases where it is impossible in arithmetic?

10. What is a convenient rule for subtracting signed numbers? State Principle VIII.

11. Write an equation whose solution is a negative number.

12. Give an example in which positive and negative numbers are multiplied. State Principle IX.

13. Define division. How do we obtain the law of signs in division? State Principle X. What is the test of the correctness of division?

14. Explain how one set of signs + and − can be used to indicate both quality and operation.

15. By means of Principles VII, VIII, IX, and X, simplify the expressions, $a+b$, $a-b$, $a \cdot b$, $\frac{a}{b}$, after substituting in each various positive and negative values of a and b.

16. Add Principles VII, VIII, IX, and X to the list which you made in Chapters I and II. It is absolutely necessary that you remember the rules stated in these principles. Any short phrases that will assist you in this are of value. For instance, the following:

VII. *In addition, positive and negative numbers tend to cancel each other. The common sign or the sign of the numerically greater is the sign of the result.*

VIII. *In subtraction, change the sign of the subtrahend and add.*

IX. *In multiplication, two like signs give + and two unlike signs give −.*

X. *In division, like signs give + and unlike signs give −.*

CHAPTER IV

ADDITION AND SUBTRACTION OF ALGEBRAIC EXPRESSIONS

54. Building Algebraic Expressions. In the preceding chapter we have noticed that in solving problems we are led to represent numbers by means of algebraic expressions which are formed by combining several algebraic symbols.

E.g. If x is a number representing my age in years, then $x - 10$ is the number representing my age 10 years ago, and $2(x - 10)$ is double the number representing my age ten years ago.

Such expressions are now to be studied more in detail.

55. Polynomials; Terms. An algebraic expression composed of parts connected by the signs $+$ and $-$ is called a **polynomial.** Each of the parts thus connected, together with the sign preceding it, is called a **term.**

E.g. $5a - 3xy - \frac{2}{3}rt + 99$ is a polynomial whose terms are $5a$, $-3xy$, $-\frac{2}{3}rt$, and $+99$. The sign $+$ is understood before $5a$.

A polynomial of two terms is called a **binomial**; one of three terms is called a **trinomial.** A term taken by itself is called a **monomial.** Terms which are to be added are called **addends.**

E.g. $5a - 3xy$ is a binomial; $5a - 3xy - \frac{2}{3}rt$ is a trinomial whose terms are the monomials $5a$, $-3xy$, $-\frac{2}{3}rt$.

According to the above definition $x + (b + c)$ may be called a binomial though it is equivalent to the trinomial $x + b + c$.

In this case x is called a **simple** term and $(b + c)$ a **compound** term. Likewise we may call $3t + 4x - 5(a + b)y$ a trinomial having the simple terms $3t$, $4x$, and the compound term $-5(a + b)y$.

56. Similar Terms. Two terms which have a factor in common are said to be **similar with respect to that factor.**

E.g. $5a$ and $-3a$ are similar with respect to a; $-3xy$ and $-7x$ are similar with respect to x; $5a$ and $-5b$ are similar with respect to 5; $7abc$ and $-\frac{2}{3}abc$ are similar with respect to abc.

Similar terms may be combined by Principle I.

E.g. $5a+3a=(5+3)a=8a$; $3xy-7x=x(3y-7)$; $5a-5b=5(a-b)$; $ax+bx-cx=(a+b-c)x$.

ORAL EXERCISES

Select the common factor and combine the similar terms in each of the following:

1. $5a+3a$.
2. $4n+5n$.
3. $2b+10b$.
4. $9c+8c+7c$.
5. $14x+7x+4x$.
6. $2k+4k+8k$,
7. $8y+3y+5y$.
8. $16z+2z+3z$.
9. $12v+4v+6v$.
10. $7m+4m+8m$.
11. $8a-3a+2a$.
12. $3x+7x-6x$.
13. $5y-2y-y$.
14. $5ab+3ab+2ab$.
15. $7x-5x+4x$.
16. $3a-2a+4a$.

WRITTEN EXERCISES

Combine similar terms in the following:

17. ax, bx, cx
18. $2ar$, $3br$, $-2cr$
19. $12cd$, $-4cf$, $-5gc$
20. $3ab$, $-2ab$, $5ab$
21. $11rs$, $-2st$, $4as$
22. $4xyz$, $2xyz$, $-3xyz$
23. $7ax$, $3bx$, $12cx$
24. $6ab$, $7ac$, $-ad$
25. $9abc$, $3dbc$, $-4ebc$
26. $5ax$, $3ax$, $2ax$
27. $4xy$, $-3yz$, $-5wy$
28. $7xyc$, $-5ayx$, $2bxy$

ADDITION AND SUBTRACTION OF POLYNOMIALS

Addition of Polynomials. In adding polynomials we use

Principle XI

57. **Rule.** *If several terms are to be added, they may be arranged and combined in any desired order.*

The truth of this principle may be seen from simple examples:

Thus, $2+3+5=3+2+5=2+(3+5)=(3+2)+5=10$.
Also, $8+(-2)+6=-2+8+6=-2+(8+6)=12$.

HISTORICAL NOTE

Associative and Commutative Laws of Addition. The fundamental character of Principle XI was first recognized about one hundred years ago. The principle as here given combines in one statement two laws of algebra: (1) **the associative law**, first so called by F. S. Servois (1814); (2) **the commutative law**, first so called by Sir William Hamilton.

The associative law states that addends may be *grouped* in any manner. Thus, $a+b+c=a+(b+c)=(a+b)+c$.

The commutative law states that addends may be put in any desired *order*. Thus, $a+b=b+a$.

58. **Arranging Terms in Columns.** In adding polynomials the work may be arranged conveniently by placing similar terms in the same column. This is permissible by Principle XI.

Example. Add $5x-6y+4z+5at$; $-3x+11y-16z-9bt$; and $-7y+8z$.

Arranging similar terms in columns, and applying Principles I and VII, we have

$$\begin{array}{r} 5x-\ 6y+\ 4z+5at \\ -3x+11y-16z-9bt \\ \quad -\ 7y+\ 8z \\ \hline 2x-\ 2y-\ 4z+(5a-9b)t \end{array}$$

Check. Putting $x=y=z=t=a=b=1$, in each of the polynomials, and in their sum we have:

$$\begin{array}{r} 5-\ 6+\ 4+5=\ \ 8 \\ -3+11-16-9=-17 \\ \quad -\ 7+8=\ \ 1 \\ \hline 2-2-\ 4+5-9=\ -8 \end{array}$$

WRITTEN EXERCISES

Arrange similar terms in columns and add:

1. Add $7b-3c+2d$; $-2b+8c-13d$.

2. Add $6x-3y+4t-7z$; $x-5y-3t$; $4x-4y+8t$.

3. Add $7a-4x+12z$; $8a-3x+2z$; $2a+4x-3z$; $5a-2x-4z$.

4. Add $5ac+3bc-14c+8b$; $2b+3c-12bc-3ac$; $4b+4c+bc-ac$; $2bc+4ac+c$; $3b-4c$.

5. Add $16xy-13cd$; $15ab-2xy$; $34cd-3xy+2ab$; $14cd-3xy-2ab$.

6. Add $34ax+4by-3z$; $2by+5z$; $3ax-7by+5z$; $7ax+4by-4z$.

7. Add $3ab+4cd-2ae$; $ab-3cd+3ae$; $3cd-2ab$; $4cd-5ae+7ab$.

8. Add $7ax-13by+5$; $9ax+8by-4$; $3by-12ax$; $4ax+7by-9$.

9. Add $5ab-3\cdot 67+5(x-1)$; $5\cdot 67+3ab-2(x-1)$; $3(x-1)-4\cdot 67+2ab$.

10. Add $11(c-9)+3(x+y)+21wu$; $-71wu-5(x+y)$; $18wu+2(x+y)-13(c-9)$.

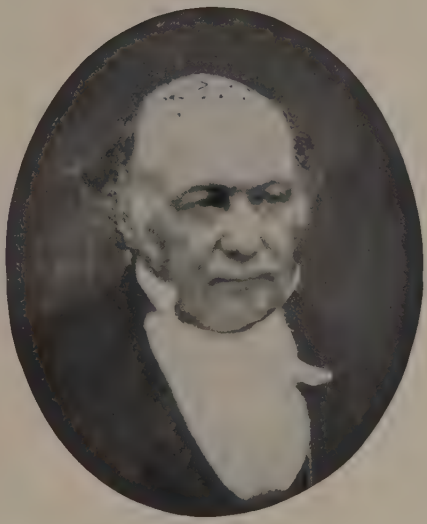
11. Add $5(a+b)-3(c-d)$; $3(c-d)-8(a+b)$; $-2(a+b)$; $13(c-d)-4(a+b)$.

12. Add $3+4(c-d)-5(a-b-c)$; $4(a-b-c)+5(c-d)$; $3(a-b-c)-9(c-d)+12$.

13. Add $(a-b)-3(c-d)+4(a+b)$; $5(a-b)+4(c-d)$; $7(c-d)-9(a-b)+3(a+b)$.

14. Add $7(x-y)-4(x+y)+4ab$; $9(x+y)+3(x-y)$; $6(x-y)-2ab-3(x+y)$.

15. Add $3(x-5)+4(b+c)+3(x-y)$; $8(b+c)-5(x-y)$; $8(x-5)-7(b+c)-4(x-y)$; $3(x-y)+(x-5)$.



Sir William Rowan Hamilton (1805–1865) was born in the city of Dublin, of Scotch parents. Already in early childhood he gave evidence of a brilliant mind. As a young man at college "amongst a number of competitors of more than ordinary merit he was first in every subject and at every examination."

Before taking his final examinations he was appointed in 1827 to the professorship of astronomy in the University of Dublin. He made a profound study of algebra and created a new branch of that subject which is called *quaternions*. In the opinion of some writers this will ultimately be regarded as one of the great discoveries of the nineteenth century.

16. Add $16(a+b-c)-3(x-y)+2(a-b)$; $2(x-y)-4xy$; $3(a-b)+(a+b-c)$; $7(a-b)+4(x-y)-8(a+b-c)$.

17. Add $6(a-b)-5(x+y)+7(x-z)-4abc$; $7(x-z)+5$; $9(x+y)+(a-b)+2abc$; $11(a-b)+10abc+3(x-z)+8(x+y)$.

18. Add $2(x-y+z)+7(a-x)-3(z-y)$; $3(x-y+z)-4(a-x)+5(z-y)$; $5(a-x)-2(z-y)+5(x-y+z)$; $5(x-y+z)+4(z-y)$.

19. Add $2ab+4(a-b)+a(b+3)+b(a+2)$; $4ab-2(a-b)+2a(b+3)-2b(a+2)$; $2b(a+2)+a(b+3)-2ab$.

20. Add $x(y-z)+y(x-z)+z(x-y)$; $5xy-2x(y-z)+3z(x-y)$; $-3xy+2x(y-z)$; $-xy-x(y-z)+2y(x-z)$.

59. **Adding Polynomials without Rewriting them.** In practice polynomials may be added without writing the similar terms in columns.

For example, to add $-3a+2b-4c$, $5a+4b+2c$, and $7a-3b-5c$, we first pick out the terms having a as a factor and add them *mentally*, then the terms containing b, and finally those containing c:

Thus, $-3a+5a+7a=9a$; $2b+4b-3b=3b$; $-4c+2c-5c=-7c$.

Hence the sum $9a+3b-7c$ may be written down at once.

ORAL EXERCISES

Pick out similar terms and add mentally:

1. $5x+3y-2z$; $-2x-7y+4z$.
2. $-3m+7n-6p$; $6m-5n+3p$.
3. $3k-7r-5s$; $9r-7k+8s$.
4. $3b+7ax+2d$; $7b-4ax+5d$.
5. $5ax+2bd-3c$; $-7ax-bd+6c$.
6. $3mn+4pq-7rs$; $-8pq+9rs-5mn$.
7. $3(a+b)+7(x-y)$; $-9(a+b)+11(x-y)$.
8. $3x-2y+4(a-b)$; $-8x+6y+8(a-b)$.
9. $4x-2y+3z$; $-5x+4y-8z$; $7x+3y-8z$.

60. Subtraction of Polynomials. Since subtraction is performed by adding the subtrahend, with its sign changed, to the minuend, we arrange the terms as in addition.

This is illustrated as follows:

From $15\,ab - 17\,xy + 11\,rt$ subtract $-5\,ab + 4\,xy - 5\,rt$.

Arranging as on page 69 and applying Principles I and VIII:

$$\begin{array}{r} 15\,ab - 17\,xy + 11\,rt \\ -5\,ab + 4\,xy - 5\,rt \\ \hline 20\,ab - 21\,xy + 16\,rt \end{array}$$

As suggested in § 45, it is sufficient to change the signs of the subtrahend *mentally*, rather than to rewrite them before adding to the minuend.

ORAL EXERCISES

Pick out the similar terms and subtract mentally.

1. From $5\,x - 3\,y + 7\,z$ subtract $2\,x + 7\,y - 9\,z$.
2. From $7\,r - 7\,s + 5\,t$ subtract $9\,r + 8\,t - 4\,s$.
3. From $6\,m - 9\,n - p$ subtract $-4\,m + 12\,n - 8\,p$.
4. From $5\,ab - 7\,bc + 11\,ac$ subtract $12\,ab - 4\,bc - 4\,ac$.
5. From $2(x-3) + 3(y-4) + 4(z-5)$
   subtract $-4(x-3) + 2(y-4) + 3(z-5)$.
6. From $5\,a - b + 2\,c - 7(x-y)$
   subtract $-11\,a - 8\,b - 5\,c + 4(x-y)$.
7. From $2\,xy - 3(x-y) + 5\,yz$
   subtract $-8\,xy - 7(x-y) - 6\,yz$
8. From $6\,ay - 3\,bz - 8\,(a-b)$
   subtract $9(a-b) + 6\,bz - 11\,ay$.
9. From $2(a+b+c) + 2\,x - 3\,y$
   subtract $6(a+b+c) - 3\,x + 5\,y$.
10. From $12(a+b-c) - 3\,ab + 2\,a - 3\,b$
    subtract $-3(a+b-c) + 6\,ab - 5\,a + 2\,b$.
11. From $5\,x - 7\,a + 8\,b + 3\,c - 9\,d$
    subtract $-3\,b + 6\,c - 6\,d - 3\,x + 3\,a$.

WRITTEN EXERCISES

Arrange similar terms in columns.

1. From $9x + 3y - 11z$ subtract $-5x + 8y - 3z$.
2. From $12ab - 3cd + 12xy$ subtract $3ab + 2cd - 11xy$.
3. From $9xc + 4ad - 3cz + 5y$ subtract $3y - 3ad + 5cz$.
4. From $13t + 5mx - 5cv$ subtract $2t - 4mx - 3cv$.
5. From $3v - 2w + 5mn - 4xz$ subtract $-v + 5w - 3mn$.
6. From $31b + 4xy + 16ax - 4$ subtract $8b - 5xy - 3ax$.
7. From $4 - 3a - 5xz - 3vy - x$ subtract $7a + 2xz + 4vy$.
8. From $8xy - 3x + 4y$ subtract $-2xy + 13w + 4x - 2y$.
9. From $2ab - 5 + 7v + 13abc$ subtract $3ab + v + 8abc$.
10. From $8acx - 4by - 3cy$ subtract $4acx + 2by + 4cy - 49$.
11. From $31 \cdot 45 - 7xy$ subtract $12 \cdot 45 + 9xy$.
12. From $3abc - 4 + 2(x + y) - 3xy$
    subtract $28 + 4xy - 3(x + y) + 8abc$.
13. From $21 + 9(xy - z) + 3(a + b)$
    subtract $8(xy - z) - 8(a + b) + 15$.
14. From $5ax - 3by + 4ax + 5by$ subtract $5by + 3ax + 7by$.
15. From $15 \cdot 48 + 8ab + 49x$ subtract $7 \cdot 48 - 9ab - 14x$.
16. From $19(r - 5s) + 13(5x - 4) + 7(x - y)$
    subtract $17(5x - 4) - 5(x - y) - 11(r - 5s)$.
17. From $30 + 14(x - 5yz) - 13(5y - z)$
    subtract $32 + 8(5y - z) - 7(x - 5yz)$.
18. From $a(b + c) + 4(m + n) - 16c$
    subtract $9(m + n) + 31c - d(b + c)$.
19. From $5(7x - 4) + 3(5y - 3x) + 35$
    subtract $56 - 9(7x - 4) + 8(5y - 3x)$.
20. From $(3a + 9b - 12c) \div 3$
    subtract $(6a - 12b - 18c) \div 6$
21. From $(axy + ayz - axz) \div a$ subtract $y(x + z) - 2xz$.
22. From $(abc + cxy) \div c$ subtract $2ab - 2xy$.

EXERCISES IN ADDITION AND SUBTRACTION

1. Add $5x-3y-7r+8t$, $-7x+18y-4r$, $-7t-20x$, $-24y+18r-15t$, and $13x+15y+11r+6t$.

Check the sum by substituting $x=1$, $y=1$, $r=1$, $t=1$.

2. Add $17a-9b$, $3c+14a$, $b-3a$, $a-17c$, $a-3b$, and $+4c$. Check for $a=1$, $b=2$, $c=3$.

3. Add $2x+3y-t$, $-6y+8t$, $-x+y-t$, $-4t+7x$, and $3y$. Check by putting each letter equal to 1.

4. Add $17r+4s-t$, $2t+3u$, $2r-3s+4t$, $5u-6t$, $7r-3s+8u$, and $8r-2t+6u$. Check as in Example 3.

5. Add $3h+2t+4u$ and $h+3t+3u$. Check by putting $h=100$, $t=10$, $u=1$; *i.e.* $324+133=457$.

6. Add $4h+3t+u$ and $3h+2t+7u$. Check as in 5.

7. Write 247, 323, 647, 239, and 41, as number expressions like those in Exs. 5 and 6 and then add them.

8. Add 647, 391, 276, and 444 as in Ex. 7.

9. Add $4t-u$, $5t-u$, $6t-u$, $7t-u$. Check for $t=10$, $u=1$.

10. Simplify: $3xyz-2xyz+5xyz-4xyz+xyz-xyz$.

11. Subtract $5a-3b+6c$ from $-8a+7b-11c$.

12. From $7xy+8xz+9yz$ subtract $17xy-19xz-20yz$.

13. From $6a-3y$ subtract $8y-3z$.

14. From $3p-4q+8r$ subtract $7p-11r+11q$.

15. From $2x-3y$ subtract $5x+7y+2a-3b$.

16. From the sum of $18abc-27xyz+13rst$ and $-11abc+16xyz-52rst$ subtract $67rst-39abc$.

17. To the difference between the subtrahend $15x-18y+27z$ and the minuend $117x+97y-81z$ add $4x-6y+3z$.

18. Add $11(x-y)+15(a-b)$ and $-20(x-y)-37(a-b)$ and from the sum subtract $135(x-y)-213(a-b)$.

ALGEBRAIC EXPRESSIONS IN PARENTHESES

61. Removing Parentheses. The sign + before parentheses means that each term within is to be added to what precedes, and the sign − means that each term within is to be subtracted from what precedes.

By Principle VII

$$a + (+b) = a + b \text{ and } a + (-b) = a - b;$$

and by Principle VIII,

$$a - (+b) = a - b \text{ and } a - (-b) = a + b.$$

Hence, we have

Principle XII

62. Rule. *Parentheses preceded by the sign + may be removed without further change.*

Parentheses preceded by the sign − may be removed by changing the sign of each term within.

Note that in each case the sign preceding the parentheses is also removed after the operation indicated by it has been carried out, and that if no sign is written before the first term in the parentheses, the sign + is understood.

Remove the parentheses and simplify the following:

Example 1. $3a + (a - b + 4) - (2a + 3b - 2)$
$= 3a + a - b + 4 - 2a - 3b + 2 = 2a - 4b + 6$

Example 2. $5(3x + y) - 4(2x - 3y + 2)$
$= 15x + 5y - 8x + 12y - 8 = 7x + 17y - 8.$

In Example 2 we multiply the terms within the first parentheses by 5 and those in the second by 4 and then remove the parentheses by Principle XII.

ORAL EXERCISES

Remove parentheses in the following:

1. $a + (x - y)$.
2. $a - (x + y)$.
3. $a - (x - y)$.
4. $-(-x - y)$.
5. $x + (a + b - c)$.
6. $x + (a - b - c)$.
7. $x - (a + b - c)$.
8. $x - (-a - b - c)$.

WRITTEN EXERCISES

Remove the symbols of aggregation and simplify:

1. $(3x-2y)-(4x+3y-2)$.
2. $x-y-2z-(3x+2y-7z)$.
3. $3(a+b+c)-2(a-b+c)$.
4. $8(5x-y+2z)-11(3x+y-z)$.
5. $5(7x-4y)+9(x-y)-3(2x+3y)$.
6. $8(r-s)+(2r+s)-(r-2s)$.
7. $11t+(2t-1)-(1-3t)$.
8. $9(r-s)-3(r+s)+2(2r-s)$.
9. $3(5x-7y)-(4x-3y+2)-5y$.
10. $5x-(8-4x+7y)+(5x+3)-(5y+3x-99)$.
11. $-(3a+5b-7c)+(8a-4c)-(9c-4b+4a)-91a$.
12. $7-(4-4c+2d-2a)+31c-(4-2a-5d)-(-8c)$.
13. $(41ab-21c+4)-(36c+15-78ab)+(13c-90ab-8)$.
14. $9by-(4c-8by-13)-2c-16-(34by-12c+8by)$.
15. $6mn+(-9m-7n+14)-8n+(13mn-17m)+34mn$.
16. $34ax-(-17ax+42)+8x-(14a+24ax-7)$.
17. $19-(2-7a-4b+11ab)-(-2b+8ab+4a)$.
18. $41by-(4b-13y+17by)-(-5b-17by+13y)$.
19. $39rs-20s-19r-(7rs+8s-19r)-(15r-5s-56)$.
20. $x-\{8x-(2y-3x)+(2x-4y)\}$.

Suggestion. First remove the parentheses, then the braces or brackets.

21. $a+[a-(b+c)-2c]$.
22. $a-\{-(a-b)+(3a-2b)\}$.
23. $2x-3(x-1)-[x-2(2x-1)]$.
24. $a-\{a+(b-c)-2(a+b+c)\}$.
25. $-\{a-[a+(b-c)-2(a+b+c)]\}$.
26. $-\{-[-(-\overline{a-b}+c-d)]\}$.
27. $-\{+[-(-\overline{a-b}-c+d)]\}$.

63. **Removing all Symbols in Order.** In such expressions as $2\,a - \{4(a-b) - (3\,a - 2\,b)\}$, the symbols of aggregation may be removed *in order as we read*, by noting that a term affected by an *even* number of minus signs is *plus*, while one affected by an *odd* number is *minus*.

Thus, in the above expression, $4\,a$ is affected by one minus sign (I), $4\,b$ by two (I and II), $3\,a$ by two (I and III), and $2\,b$ by three (I, III, and IV). Hence the expression equals $2\,a - 4\,a + 4\,b + 3\,a - 2\,b = a + 2\,b$.

ORAL EXERCISES

Remove all symbols of aggregation *as you read*, then write the result, if necessary, and simplify.

1. $a - (2\,a + 3\,b)$.
2. $x - [2\,x - (x + y)]$.
3. $m - [3(m - n) - 4\,m]$.
4. $r + (3\,a - 2\,b) - (a - b)$.
5. $a - \{2\,a - (3\,a - b)\}$.
6. $a + \{2\,a + (3\,a - b)\}$.
7. $a + \{2\,a - (3\,a - b)\}$.
8. $a - \{2\,a + (3\,a - b)\}$.
9. $a - \{2\,a - (3\,a + b)\}$.
10. $a - \{2\,a - [3\,a - (a + b)]\}$.

11. Read Examples 20 to 27 in the preceding exercises in this manner.

64. **Inserting Expressions in Parentheses.** By the converse of Principle XII terms may be inclosed in parentheses with or without change of sign, according as the sign $-$ or $+$ precedes.

E.g. $a + b - c = a + (b - c)$ and $a - b + c = a - (b - c)$.

ORAL EXERCISES

In each of the following, place the last two terms in parentheses.

1. $x - y + 2$.
2. $5\,a + 3\,b - c$.
3. $m - n + p$.
4. $5\,a - 3\,b + 2\,c$.
5. $7\,m - 4\,n - 3\,p$.
6. $8 + 4\,b - 3\,c$.
7. $5\,xy - 5\,x + 3\,y$.
8. $9\,ax + 3\,by - 4\,cd$.
9. $a - 3\,b + d - 5\,c$.
10. $13 - 7\,a - 3\,b$.
11. $19\,x - 3\,c + 4\,e$.
12. $21\,ax - 13\,bx + 6\,dx$.

WRITTEN EXERCISES

In the following, use either method: (1) remove one symbol at a time, beginning with the innermost; (2) remove them in order as you write, beginning at the left.

1. $-[a+\{a-(a-x)-(a-\overline{x-a})\}-a]$.
2. $8x-\{6y-[3x-(2y-x)-8y]+x\}$.
3. $-[x-\{z+(x-z)-(z-x)-z\}-x]$.
4. $2a-[2a-\{2a-(2a-\overline{2a-a})\}]$.
5. $-[a-\{a-(x-\overline{a-x-a})-a\}-2a]$.
6. $-[5x-\{4y-(5z+2y)-(2x-5z)\}]$.
7. $16-x-[7x-\{8x-(9x-\overline{3x-6x})\}]$.
8. $2x-[3y-\{4x-(5y-\overline{6x-7y})\}]$.
9. $2a-[3b+(2b-c)-4c+\{2a-(3b-\overline{c-2b})\}]$.
10. $a-[5b-\{a-(5c-\overline{2c-b}-4b)+2a-(a-\overline{2b+c})\}]$.
11. $2(3b-5a)-7[a-6\{2-5(a-b)\}]$.
12. $-2\{a-6[a-(b-c)]\}+6\{b-(c+a)\}$.
13. $-3\{-2[-4(-a)]\}+5\}-2[-2(-a)]\}$.
14. $-2\{-[-(x-y)]\}+\{-2[-(x-y)]\}$.
15. $a-(b-c)-[a-b-c-2\{b+c-3(c-a)-d\}]$.
16. $2x-(3y-4z)-\{2x-[3y+4z-3y-(4z+2x)]\}$.
17. $-2(a-d)-2[b+c+d-3\{c+d-4(d-a)\}]$.
18. $-4(a+d)+4(b-c)-2[c+d+a-3\{d+a-4(b+c)\}]$.
19. $a-2b-[4a-6b-\{3a-c+(5a-2b-\overline{3a-c+2b})\}]$.
20. $a-[-b\{-c(-d+\overline{e-f})+2a-c\}+c+d]$.

In each of the following insert the last three terms in parentheses preceded by the minus sign:

21. $5+2a-3+b$.
22. $x-4-2a+c$.
23. $7xy-3x-2y+z$.
24. $a-2b-4c+d$.
25. $6-a-b-c$.
26. $3b-4-2x+y$.
27. $9y-3c+8-2a$.
28. $12+3a-8b+c$.
29. $c-8b-3d+12$.
30. $3a-4c+5d-2b$.

FORMING ALGEBRAIC EXPRESSIONS

ORAL EXERCISES

1. The sum of two numbers is 20. If x is one of them, how may the other be represented?

2. The sum of two numbers is 16. What is a convenient representation of each number?

3. How do you represent six times the number x?

4. How do you represent 6 less than twice the number n?

5. How do you represent 8 more than 4 times the number a?

6. If x is a number, how do you represent a number which is 8 more than $\frac{1}{3}$ of this number?

7. If a is a number, how do you represent a number 10 less than $\frac{3}{4}$ of this number?

8. A number is represented by x, another number is 15 less than twice this number. How do you represent the sum of these numbers?

9. A number is represented by n. Another number is 6 more than three times this number. How do you represent twice the second number?

10. A number is represented by x. Another number is 24 less than four times this number. How do you represent $\frac{1}{3}$ of the second number?

11. A number is represented by a. By how much does this number exceed 50?

12. The difference between two numbers is 10. How may we represent the numbers?

13. If a number exceeds another by 50, represent twice the smaller number plus three times the greater.

14. If n represents the number of years in my age now, how old was I five years ago? How old will I be 10 years hence?

PROBLEMS ON THE RELATIONS OF NUMBERS

1. The sum of two numbers is 20 and one is 4 greater than the other. Find the numbers.

2. If 10 is added to 4 times a number, the result is 74. Find the number.

3. Six less than twice a certain number equals 18. Find the number.

4. Eight more than 4 times a number equals 48. Find the number.

5. Ten more than $\frac{1}{2}$ of a certain number equals 18. Find the number.

6. One number is 12 greater than another. Their sum is 24. Find the numbers.

7. One number is 10 less than twice another number. The sum of the numbers is 50. Find the numbers.

8. One number is 18 more than 3 times another number. The sum of the numbers is 66. Find the numbers.

9. Find two numbers such that one is 6 more than twice the other, and the smaller plus twice the greater equals 42.

10. One number exceeds another by 50. Twice the smaller number plus the larger number equals 95. Find the numbers.

11. The difference between two numbers is 10. The smaller number plus twice the larger equals 35. Find the numbers.

12. One number exceeds another by 40. Twice the smaller number plus 3 times the greater equals 295. Find the numbers.

13. One number is 16 less than 6 times another. The sum of the numbers is 40. Find the numbers.

14. The sum of my ages 10 years hence and 5 years ago is 55. How old am I?

PROBLEMS ON THE ARRANGEMENT AND VALUE OF DIGITS

If we speak of the number whose 3 digits, in order from left to right, are 5, 3, and 8, we mean $538 = 500 + 30 + 8$. Likewise, the number whose three digits are h, t, and u is written $100\,h + 10\,t + u$. Note that htu would mean $h \times t \times u$.

Hence, when letters stand for the digits of a number written in the decimal notation, care must be taken to multiply each letter by 10, 100, 1000, etc., according to the position it occupies.

ORAL EXERCISES

1. If the sum of two digits is 9 and if x represents one of them, how do you represent the other?

2. If the difference between two digits is 3, and if x represents one of the digits, how do you represent the other?

3. Give an expression representing a number if the digit in tens' place is x and the digit in units' place is y. Also give an expression representing the number if the order of the digits is reversed.

4. Give an expression representing a number if the digit in tens' place is x and in units' place $6 - x$. Also give an expression representing the number with the order of the digits reversed.

5. Give an expression representing a number if the digit in tens' place is t and that in units' place $t + 6$. Also give an expression representing the number obtained by reversing the order of the digits.

6. If the digit in tens' place is x and in units' place $12 - x$, give an expression representing 7 times the sum of the digits.

7. If the digit in tens' place is x and the digit in units' place is 12 less than 5 times x, write an expression representing the number. Also express it with the digits reversed.

Illustrative Problem. A number is composed of two digits whose sum is 6. If the order of the digits is reversed, we obtain a number which is 18 greater than the first number. What is the number?

Solution. Let $x =$ the digit in tens' place.

Then, $6 - x =$ the digit in units' place.

Hence, the number is $10x + 6 - x$. Reversing the order of the digits, we have as the new number $10(6 - x) + x$.

Hence, $10(6 - x) + x = 18 + 10x + 6 - x$.

Solving, $x = 2$, the digit in ten's place.

and $6 - x = 4$, the digit in unit's place.

Hence, the required number is 24.

WRITTEN EXERCISES

In each of the examples 1 to 8 below the number considered is composed of two digits.

1. The digit in units' place is 2 greater than the digit in tens' place. If 4 is added to the number, the result is then equal to 5 times the sum of the digits. What is the number?

2. The digit in tens' place is 3 greater than the digit in units' place. The number is 1 more than 8 times the sum of the digits. What is the number?

3. The sum of the digits is 9. If the order of the digits is reversed, we obtain a number which is equal to 12 times the remainder when the units' digit is taken from the tens' digit. What is the number?

4. The sum of the digits is 12. If the order of digits is reversed, the number is increased by 18. Find the number.

5. The tens' digit is 2 less than its units' digit. The number is 1 less than 5 times the sum of its digits. What is the number?

6. The digit in units' place is 4 less than that in tens' place. If the order of the digits is reversed, we obtain a number which is 3 less than 4 times the sum of the digits. What is the number?

7. The digit in units' place is 2 less than twice the digit in tens' place. If the order of the digits is reversed, the number is unchanged. What is the number?

8. The digit in tens' place is 12 less than 5 times the digit in units' place. If the order of the digits is reversed, the number is equal to 4 times the sum of the digits. What is the number?

9. A number is composed of three digits. The digit in units' place is 3 greater than the digit in tens' place, which in turn is 2 greater than the digit in hundreds' place. The number is equal to 96 plus 4 times the sum of the digits. What is the number?

REVIEW QUESTIONS

1. What is a *polynomial?* a *term?* How are polynomials classified? What are *similar terms?* By what principle are similar terms added? By what principle are they subtracted?

2. In adding or subtracting polynomials how may the terms be arranged for convenience? State the principle on which this is based.

3. What is the rule for removing parentheses when preceded by the sign $+$? By the sign $-$? How may Principle XII be used for inclosing terms within parentheses?

4 Tell how to remove all the symbols of aggregation in order as you read $5x-\{x-(3x+y)-(2x-3y)\}$. How many minus signs affect the term $3x$? How many affect $3y$? What are the final signs of these terms?

5. Add Principles XI and XII to your list expressed in symbols:

XI. $a+b+c=a+c+b=c+a+b$, etc.

XII. $a+(b-c-d)=a+b-c-d.$

$a-(b-c-d)=a-b+c+d$

Hand from here on.

$x + x = x^2$

CHAPTER V

MULTIPLICATION AND DIVISION OF ALGEBRAIC EXPRESSIONS

MULTIPLICATION OF POLYNOMIALS

In **multiplying one monomial by another** we use

Principle XIII

65. **Rule.** *To obtain the product of two or more factors, these may be arranged and multiplied in any order.*

The truth of this principle is clear from examples such as:

$$2 \cdot 3 \cdot 5 = 2 \cdot 5 \cdot 3 = 5 \cdot 3 \cdot 2 = 5 \cdot (3 \cdot 2) = 2 \cdot (3 \cdot 5) = 30.$$

HISTORICAL NOTE

Associative and Commutative Laws of Factors. Principle XIII, like Principle XI, states two fundamental laws of algebra: (1) **the associative law of factors,** first so called by F. S. Servois; (2) **the commutative law of factors,** first so called by Sir William Hamilton.

The associative law of factors states that factors may be grouped in any combination.

Thus, $abc = a(bc) = (ab)c.$

The commutative law of factors states that factors may be multiplied together in any order.

Thus, $abc = acb = cba,$ etc.

66. **Repeated Factors.** In multiplying algebraic expressions, the same factor may occur more than once in the same term.

Thus we may have $5 \cdot 5$ or $x \cdot x$. These are written 5^2 and x^2 respectively, and are read *5 square* and *x square*.

This is a convenient way of abbreviating written expressions.

E.g. $5a \cdot 3a = (5 \cdot 3) \cdot (a \cdot a) = 15a^2;\ ay \cdot ay = aayy = a^2y^2.$

67. The product of two binomials such as $5+8$ and $5+3$ may be obtained in two ways:

(1) $(5+3)(5+8) = 8 \cdot 13 = 104.$

(2) $(5+3)(5+8) = 5(5+8)+3(5+8) = 5^2+5 \cdot 8+3 \cdot 5+3 \cdot 8 = 104.$

The second method is illustrated by the accompanying figure in which $5+8$ is the length of a rectangle and $5+3$ is its width. The total area is the product $(5+3) \cdot (5+8)$ and it is composed of the four small areas, 5^2, $5 \cdot 8$, $3 \cdot 8$, and $3 \cdot 5$.

	5	8
3	$3 \cdot 5$	$3 \cdot 8$
5	$5 \cdot 5$	$5 \cdot 8$
	5	8

The second method here used for multiplying $(5+3)(5+8)$ is the only one available when the terms of the binomials cannot be combined.

Thus, $(x+4)(x+6) = x(x+6)+4(x+6) = x^2+6x+4x+4 \cdot 6$
$= x^2+10x+24,$

and $(a+b)(m+n) = a(m+n)+b(m+n) = am+an+bm+bn.$

Hence, **to multiply two binomials,** *multiply each term of one by every term of the other and add the products.*

68. Product of Two Trinomials. In a manner similar to that just illustrated we may multiply two trinomials.

	m	n	r
a	am	an	ar
b	bm	bn	br
c	cm	cn	cr

From the figure we see at once that

$(a+b+c)(m+n+r) = am+bm+cm+an+bn+cn+ar+br+cr,$

in which *each term of one trinomial is multiplied by every term of the other and the products are added.*

Evidently the same process is applicable to the product of two such polynomials each containing any number of terms.

WRITTEN EXERCISES

Find the following products:

1. $(x+1)(x+2)$.
2. $(x+3)(x+5)$.
3. $(u+7)(u+4)$.
4. $(a+8)(a+3)$.
5. $(t+3)(t+7)$.
6. $(y+9)(y+2)$.
7. $(x+1)(x+7)$.
8. $(s+5)(s+3)$.
9. $(a+b)(c+d)$.
10. $(x+4)(x+3)$.
11. $(x+y)(a+b)$.
12. $(2x+3)(x+2)$.
13. $(5+x)(6+x)$.
14. $(5s+1)(s+5)$.
15. $(x+7)(3x+4)$.
16. $(a+4)(3a+1)$.
17. $(3+x)(2+5x)$.
18. $(a+b)(3a+7b)$.
19. $(x+y)(2x+3y)$.
20. $(7x+4)(x+8)$.
21. $(x+3)(2x+3)$.
22. $(2a+5)(a+7)$.
23. $(8b+3)(2b+3)$.
24. $(5a+4)(2a+3)$.
25. $(9+2x)(5+x)$.
26. $(6+y)(3+4y)$.

Example. Solve the equation

$$(x+1)(x+2) = x(1+x)+12. \tag{1}$$

Solution. Performing the indicated multiplication,

$$x^2+3x+2 = x+x^2+12 \tag{2}$$

Subtracting x^2 from both sides,

$$3x+2 = x+12. \tag{3}$$

By $S \mid x$, 2, $\quad 2x = 10. \tag{4}$

By $D \mid 2$, $\quad x = 5. \tag{5}$

Equation (2) differs from those solved heretofore in that it contains the term x^2 in each member.

We may, however, subtract x^2 from each member, giving equation (3), which is a form already studied.

Solve the following equations:

27. $(x+3)(x+2) = (x+1)(x+2)+6$.
28. $(x+2)(x+4) = (x+3)(x+1)+9$.
29. $(x+3)(x+5)-13 = (x+2)(x+4)$.
30. $(x+1)(x+7)-2 = (x+2)(x+5)$.

WRITTEN PROBLEMS

1. The length of a rectangle is 6 feet greater than its width w. Express the area of the rectangle in terms of w.

2. The width w of a rectangle is 10 feet less than its length. Express the area of the rectangle in terms of w.

3. The length of a rectangle is 4 greater than its width w. Express the dimensions of this rectangle in terms of w after the width is increased by 2 and the length by 3.

4. The width w of a rectangle is 8 less than its length. Express the dimensions of this rectangle in terms of w after its length is increased by 2 and its width is increased by 4. Also express the area of the new rectangle in terms of w.

5. A field is 10 rods longer than it is wide. If its length is increased by 10 rods and its width increased by 5 rods, the area is increased by 640 square rods. What are the dimensions of the field?

Suggestions. Let $w =$ the width of the field.
Then, $w + 10 =$ its length,
and $w(w + 10) =$ its area.
The area of the increased field is $(w + 5)(w + 20)$.
By the conditions of the problem,

$$(w + 5)(w + 20) = w(w + 10) + 640.$$

6. A rectangle is 7 feet longer than it is wide. If its length is increased by 3 feet and its width is increased by 2 feet, its area is increased by 60 square feet. What are its dimensions?

7. A farmer has a plan for a granary which is to be 12 feet longer than wide. He finds that if the length is increased 8 feet and the width is increased 2 feet, the floor space will be increased by 160 square feet. What are the dimensions?

8. If the length of a rectangular flower bed is increased 3 feet and its width is increased 1 foot, its area will be increased by 19 square feet. What are its present dimensions, if its length is 4 feet greater than its width?

69. **Multiplying Polynomials with Negative Terms.** In §§ 67 and 68 we studied the multiplication of polynomials whose terms were all positive. The same method may be applied to polynomials having negative terms.

Example. Find the product of $(5-2)$ and $(4-3)$. This product, written out term by term as a product of two sums, would give

$$\begin{aligned}(5-2)(4-3) &= [5+(-2)][4+(-3)] \\ &= 5\cdot 4 + 5(-3) + 4(-2) + (-3)(-2) \\ &= 20 - 15 - 8 + 6 = 3.\end{aligned}$$

Also $(5-2)(4-3) = 3\cdot 1 = 3$.

Similarly, $(x+5)(x-2) = x^2 + 5x - 2x - 10 = x^2 + 3x - 10$,

and $(x-3)(x-5) = x^2 - 3x - 5x + 15 = x^2 - 8x + 15$.

WRITTEN EXERCISES

Perform the following indicated operations:

1. $(x-5)(x-3)$.
2. $(x-3)(x+4)$.
3. $(a-6)(a-1)$.
4. $(u+5)(u-3)$.
5. $(b+2)(b-7)$.
6. $(3-b)(4+b)$.
7. $(3+x)(7-3x)$.
8. $(n-4)(3-n)$.
9. $(a-b)(c+d)$.
10. $(a-b)(c-d)$.
11. $(x-4)(x-5)$.
12. $(a+b-c)(m-n)$.
13. $(a-b)(7a+3b)$.
14. $(5-y)(5x+3y)$.
15. $(2a-3b+c)(m+n)$.
16. $(v-t)(7v-5t)$.
17. $(3a-2)(2b-3)$.
18. $(7x-3y-4)(a-4b)$.
19. $(6+3a-b)(4c-9d-1)$.
20. $(a+m+n)(x-y+z)$.
21. $(a+b-c)(d-e+f)$.
22. $(v+t+u)(v-t-u)$.
23. $(3x-5)(2x+7)$.
24. $(2x-y-1)(x+y)$.
25. $(8a-3b-1)(2a-b)$.
26. $(2x-1+y)(3x-2y)$.
27. $(4a-b)(a+2b)$.
28. $(6x+5y)(2x-y)$.
29. $(7m-3n)(2m+5n)$.
30. $(8a-3b)(2a+3b)$.
31. $(5c+d)(2c-3d)$.
32. $(3a-4b)(3a+4b)$.
33. $(9x-4y)(4x+9y)$.
34. $(12a-5b)(3a-b)$.

The preceding exercises illustrate

Principle XIV

70. **Rule.** *The product of two polynomials is found by multiplying each term of one by every term of the other, and adding the products.*

71. It should be observed that Principle XIV involves a repeated application of Principle II. Thus

$$(a+b)(c+d)=(a+b)c+(a+b)d=ac+bc+ad+bd.$$

Arranging the Terms of the Product. In multiplying polynomials, the work is usually arranged so that similar terms in the product are written in columns and then added.

Example 1. Multiply $3x-2$ by $2x-5$.

Solution.

$$\begin{array}{r} 3x-2 \\ \underline{2x-5} \\ 6x^2-4x \\ \underline{\quad -15x+10} \\ 6x^2-19x+10 \end{array}$$

Example 2. Multiply $3x-2y+2$ by $4x-3y-2$.

Solution.

$$\begin{array}{r} 3x-2y+2 \\ \underline{4x-3y-2} \\ 12x^2-8xy+8x \\ \quad -9xy \quad +6y^2-6y \\ \underline{\quad -6x \quad +4y-4} \\ 12x^2-17xy+2x+6y^2-2y-4 \end{array}$$

WRITTEN EXERCISES

In each case simplify the expressions within the parentheses as much as possible before multiplying:

1. $(x-7)(3x+4)$.
2. $(x-2)(9x+4)$.
3. $(a-x)(9x+4a)$.
4. $(11b+3a)(2b-3b+5)$.
5. $(x-2+y)(4y-3x)$.
6. $(x-5x+4)(8y-3-y)$.
7. $(7+y-x)(2y+x-1)$.
8. $(5x+3y-1)(x-2)$.
9. $(x-y+3)(5x-3y+5)$.
10. $(a-13n)(a-n+8)$.

write out 5 probs. marked.

WRITTEN EXERCISES AND PROBLEMS

Multiply the following:

1. $(13a - b - 12a)(2b - 3a)$.
2. $(6 - 4x + 3x)(7x + y - 3x + 1)$.
3. $(5x + 3y - 4x - 2y)(6y + 3x - 2y + y)$.
4. $(11b - a - 10b)(6a - 3b - 2a)$.
5. $(-7a - 1 + 8a)(5a - 8 - 3a)$.

Solve the following equations and check the results:

6. $(x + 2)(x + 3) = (x - 3)(x + 10) + 10$.
7. $(5x - 4)(6 - x) - 97 = (x - 1)(6 - 5x)$.
8. $(3n - 1)(18 - n) = (n + 6)(16 - 3n)$.
9. $(7 - a)(9a - 8) = 31 + (36 - 9a)(a + 2)$.
10. $(4a + 4)(a - 3) = (4a + 1)(a + 7) - 13a + 221$.
11. $(n + 6)(3n - 4) - 14 = (n + 8)(3n - 3)$.
12. $(8n + 6)(10 - n) + 150 = (1 - n)(8n + 3)$.
13. $(a - 1)(13 - 6a) = (6a - 3)(8 - a) - 21$.
14. $(7x - 13)(6 - x) - (x + 4)(3 - 7x) = 70$.

In the following make sure that the solutions are correct by doing the work with care and looking it over a second time.

15. $(2x - 3)(5x + 2) - 7x = (2 - 5x)(7 - 2x) + 1$.
16. $(2a - 1)(3a + 2) + a = (4 - 3a)(1 - 2a) + 7$.
17. $(8x - 1)(x + 7) + 4x = (2x + 8)(4x + 6) + 5$.
18. $(5b + 2)(3 - b) = (3 - 5b)(4 + b) + 9$.
19. $(3a - 7)(2 + a) = (5 + 3a)(2 + a)$.
20. $(x - 3)(2x + 5) - (3 - x)(5 - 2x) = 0$.
21. $(3y - 5)(5 - 6y) - (9y - 6)(3 - 2y) = 5$.
22. $(4 - a)(3 - a) + (a - 2)(a - 5) + (1 - 2a)(2 + a) = 7$.
23. Find two numbers whose difference is 6 and whose product is 180 greater than the square of the smaller.

24. There are four consecutive *even* integers such that the product of the first and second is 40 less than the product of the third and fourth. What are the numbers?

25. There are four consecutive integers such that the product of the first and third is 223 less than the product of the second and fourth. What are the numbers?

26. Find four numbers such that the second is 5 greater than the first, the third 5 greater than the second, and the fourth 5 greater than the third. The product of the first and second is 250 less than the product of the third and fourth.

27. A club makes an equal assessment on its members each year to raise a certain fixed sum. One year each member pays a number of dollars equal to the number of members of the club less 175. The following year, when the club has 50 more members, each member pays \$5 less than the preceding year. What was the membership of the club the first year and how much did each pay?

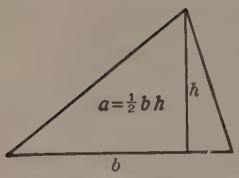
PROBLEMS ON RECTANGLES AND TRIANGLES

28. A rectangle is 10 inches longer than wide. Express its area in terms of the width w. If the width is increased by 4 and the length by 6 inches, express the area in terms of w.

29. A rectangle is 8 inches longer than wide. Express its area in terms of the width w, after the width is increased 4 inches and the length decreased 10 inches.

30. A rectangle is 5 feet longer than it is wide. If it were 3 feet wider and 2 feet shorter, it would contain 15 square feet more. Find the dimensions of the rectangle.

31. A rectangle is 6 feet longer and 4 feet narrower than a square of equal area. Find the side of the square and the sides of the rectangle.



If b is the base of a triangle, h its height or altitude, and a its area, then *area* $= \frac{1}{2}$ (*base* $\times$ *altitude*);

i.e. $a = \frac{1}{2}bh$.

32. The base of a triangle is 2 inches less than its altitude a. Express the area in terms of a.

33. The altitude of a triangle is 7 greater than its base b. If the altitude is increased by 8 and the base by 6, express its area in terms of b.

34. The altitude of a triangle is 16 inches less than the base. If the altitude is increased by 3 inches and the base by 2 inches, the area is increased by 52 square inches. Find the base and altitude of the triangle.

PRODUCTS OF POWERS OF THE SAME BASE

72. Exponents; Powers. Any number written over and to the right of a number expression is called an **index** or **exponent**. If an exponent is a *positive integer*, it shows how many times the expression is to be taken as a factor.

A product consisting entirely of equal factors is called a **power** of the repeated factor. The repeated factor is called the **base** of the power. See § 66.

E.g. x^3 means $x \cdot x \cdot x$ and is read *the third power of* x or x *cube*; x^5 means $x \cdot x \cdot x \cdot x \cdot x$, and is read *the fifth power of* x or briefly x *fifth*. In both these cases the base is x. $(x-y)^3 = (x-y)(x-y)(x-y)$ and is read $x-y$ *cubed* or *the cube of the binomial* $x-y$.

The first power of x is written without an exponent. Thus x means x^1; 2 means 2^1, etc.

Difference between a Coefficient and an Exponent. A coefficient *is a factor*, while an exponent, if a positive integer, shows how many times some number is *used as a factor*.

E.g. $5a = 5 \cdot a$ which means $a + a + a + a + a$, while $a^5 = a \cdot a \cdot a \cdot a \cdot a$.

ORAL EXERCISES

Find the following powers:

1. 2^3, 2^4, 2^5.
2. 3^2, 3^3.
3. 4^2, 4^3.
4. 5^2, 5^3.
5. 6^2, 6^3.
6. 7^2, 8^2.
7. 9^2, 10^2.
8. 11^2, 12^2.
9. 13^2, 14^2.
10. 15^2, 16^2.
11. 20^2, 30^2.
12. 40^2, 50^2.
13. 60^2, 70^2.
14. 80^2, 90^2.
15. 100^2, 1000^2.
16. 200^2, 300^2.
17. 400^2, 500^2.
18. 600^2, 700^2.

WRITTEN EXERCISES

Find the following powers:

1. 17^2, 18^2.
2. 19^2, 21^2.
3. 22^2, 23^2.
4. 24^2, 25^2.
5. 26^2, 27^2.
6. 28^2, 29^2.
7. $(a+b)^2$.
8. $(c-d)^2$.
9. $(a+b+c)^2$.
10. $(a+b-c)^2$.
11. $(3-a)^2$.
12. $(3-b-c)^2$.
13. $(a-b-c)^2$.
14. $(3a-2)^2$.
15. $(x-y+z)^2$.
16. $(2x-3y)^2$.
17. $(5a-2b)^2$.
18. $(4x+3y)^2$.

73. In the case of factors expressed in Arabic numerals multiplications like the following may be carried out in either of two ways.

E.g. $3^2 \cdot 3^4 = 9 \cdot 81 = 729,$

or $3^2 \cdot 3^4 = (3 \cdot 3)(3 \cdot 3 \cdot 3 \cdot 3) = 3^{2+4} = 3^6 = 729.$

But with literal factors the second process only is possible.

E.g. $a^2 \cdot a^4 = (a \cdot a)(a \cdot a \cdot a \cdot a) = a^{2+4} = a^6.$

WRITTEN EXERCISES

In the following exercises carry out each indicated multiplication in two ways in case this is possible:

1. $5 \cdot 5^2$.
2. $5^2 \cdot 5^4$.
3. $3^2 \cdot 3^3$.
4. $7 \cdot 7^3$.
5. $a^2 \cdot a^3$.
6. $x^3 \cdot x^2$.
7. $x^7 \cdot x^4$.
8. $t^3 \cdot t^4$.
9. $t^2 \cdot t^3 \cdot t^4$.
10. $2^3 \cdot 2^2 \cdot 2^4$.
11. $3 \cdot 3^2 \cdot 3^3$.
12. $2^2 \cdot 2^3 \cdot 2^2 \cdot 2$.

Illustrative Problem. To multiply 2^k by 2^n, k and n being any two positive integers.

Solution. 2^k means $2 \cdot 2 \cdot 2 \cdot 2$, etc., to k factors,
and 2^n means $2 \cdot 2 \cdot 2 \cdot 2$, etc., to n factors.
Hence, $2^k \cdot 2^n = (2 \cdot 2 \cdot 2 \cdots$ to k factors$)\ (2 \cdot 2 \cdots$ to n factors$)$
$= 2 \cdot 2 \cdot 2 \cdot 2 \cdots$ to $k + n$ factors in all.
That is, $2^k \cdot 2^n = 2^{k+n}$.

The preceding examples illustrate

Principle XV

74. Rule. *The product of two powers of the same base is found by adding the exponents of the factors and making this sum the exponent of the common base.*

But exponents are added in multiplication only when they apply to the *same* base. Thus $2^2 \cdot 2^3 = 2^{2+3} = 2^5 = 32$; $a^2 \cdot a^3 = a^5$.

E.g. The product of $2^3 \cdot 3^2$ cannot be found by adding the exponents. It must be done as follows: $2^3 \cdot 3^2 = 8 \cdot 9 = 72$.

ORAL EXERCISES

Perform the following indicated multiplications by means of Principle XV.

1. $2^3 \cdot 2^7$.	7. $4^a \cdot 4^b$.	13. $x^2 \cdot 3\,x^3$.
2. $a^3 \cdot a^7$.	8. $3^{2a} \cdot 3^{2b}$.	14. $x^4 \cdot 4\,x^5$.
3. $3^4 \cdot 3^5$.	9. $5^{2+n} \cdot 5^{2-n}$.	15. $x^3 \cdot 5\,x^5$.
4. $x^4 \cdot x^5$.	10. $a^m \cdot a^n$.	16. $a^k \cdot 2\,a^n$.
5. $3^k \cdot 3^n$.	11. $c^x \cdot c^{2-x}$.	17. $a^4 \cdot 2\,a^n$.
6. $x^k \cdot x^n$.	12. $x^{ac} \cdot x^{bc}$.	18. $a^{2n} \cdot 3\,a^{3n}$.

Perform the following multiplications by means of Principles II and XV.

19. $x(x^2 + x + 1)$.	23. $y(3\,y^2 + 4\,y^4 - y^3)$.
20. $x^2(x + 1)$.	24. $x^3(7\,x^4 - 5\,x^3 - 2\,x)$.
21. $a^2(a^2 - a + 1)$.	25. $a^3(3\,a - 2\,a^2b + 4\,ab^2)$.
22. $a^3(a^4 - 4\,a^2 + a + 1)$.	26. $a^{2b}(4a^{2b} - a^{3c} + a^{2d})$.

75. **Products of Monomials.** In multiplying together monomials like $3\,a^2bc$ and $2\,ab^2cd$ it is convenient to arrange the factors in the product so that the same letters are associated together and likewise the numerical coefficients. This we are permitted to do by Principle XIII.

Thus, $3\,a^2bc \times 2\,ab^2cd = (3 \cdot 2)(a^2 \cdot a)(b \cdot b^2)(c \cdot c)d = 6\,a^3b^3c^2d$.

Notice that in the product *the exponent of each letter is the sum of the exponents of this letter in the factors, and the numerical coefficient is the product of the numerical coefficients of the factors.*

E.g. $(2\,a^2b^2)(5\,a^4b^3c) = 10\,a^{4+2}b^{2+3}c = 10\,a^6b^5c$.

This is a convenient rule for finding the product of two or more monomials.

ORAL EXERCISES

Multiply:

1. $3\,ab$ by $5\,a^2b^2$.
2. $4\,x^3$ by $3\,xy^2$.
3. $2\,xyz$ by $3\,x^2yz$.
4. $6\,k^2$ by $7\,ak^3$.
5. $3\,x^4y^2$ by $4\,xy^3$.
6. $5\,a^2b^3c$ by ab^4c.
7. $2\,b^4c^2x^3$ by $5\,b^4cx$.
8. $3\,a^3b^4c$ by abd^4.
9. $-2\,x^2y^2$ by $5\,xy^3$.
10. $-4\,a^2bc^3$ by $-3\,ab^3c^2$.
11. $-3\,t^5u^6$ by $-5\,tu^4$.
12. a^2x^2 by $-3\,ax^5$.
13. $5\,abc$ by $-abc$.
14. $-4\,m^2n$ by $-6\,mn^2$.
15. $-7\,a^2my$ by $-2\,m^2y$.
16. $-3\,x^3y^2$ by $2\,xy^4$.
17. $10^2x^2y^2$ by $-10^3\,x^3y$.
18. $5^3 \cdot 3^2 \cdot 2^2$ by $5 \cdot 3 \cdot 2^4$.
19. $2\,a^3bc^3$ by $3\,ab^4c$.
20. $6\,m^2n$ by $3\,mn^3$.
21. $2\,ma^5$ by $3\,a^2$.
22. $2\,b^4$ by $7\,b^4$.
23. $4\,x^my^2$ by $3\,x^ny^5$.
24. $3\,my^2$ by $4\,m^2y^3$.
25. $7\,ax^3$ by $4\,a^3x^2$.
26. $5\,a^4y$ by a^2y^3.
27. $4\,m^2n^3$ by m^4n^2.
28. $-3\,a^3b^4c$ by $5\,a^2b^4c^7$.
29. $-8\,a^3c^5$ by $-4\,a^7c^2x$
30. $-4\,x^4y^2$ by $3\,x^2y^4$.

WRITTEN EXERCISES

Perform the following indicated multiplications:

1. $(a+b)(a^2+b^2)$.
2. $(a-b)(a^2+b^2)$.
3. $(a-b)(a^2-b^2)$.
4. $(x^2+y^2)(x^2-y^2)$.
5. $(x-y^2)(x^2-y)$.
6. $(3\,a^2x-y^2)(3\,a^2x+y^2)$.
7. $(3\,x^2-2\,y^2)(3\,x^2+2\,y^2)$.
8. $(2\,ax^2-3\,by^2)(2\,ax^2+3\,by^2)$.
9. $(7\,x^2+2\,y)(3\,x^2-2\,y)$.
10. $(2\,a^2b-c^2)(3\,ab^2+c^2)$.
11. $(3\,xy^2-5\,x^2y)(x-y)$.
12. $(a+b)^2(a-b)$.
13. $(a-b)^2(a+b)$.
14. $(a+b)^2(a-b)^2$.
15. $(a^3+a^2b+ab^2+b^3)(a-b)$.
16. $(a+b-c)(a+b+c)$.
17. $(3\,x-2\,y-1)(2\,x+y)$.
18. $(1+a+a^2)(1-a)$.
19. $(1-a+a^2-a^3)(1+a)$.
20. $(a+b)(a-b)(a^2+b^2)$.
21. $(a+b)(a^2-ab+b^2)$.
22. $(a-b)(a^2+ab+b^2)$.
23. $(x^2+x+1)(x^2-x+1)$.
24. $(x+y)(x^3-x^2y+xy^2-y^3)$.
25. $(x-y)(x^3+x^2y+xy^2+y^3)$.
26. $(x^2-xy+y^2)(x^2+xy+y^2)$.
27. $(x^2+2\,xy+y^2)(x-y)^2$.
28. $(x^2-y^2)(x^4+x^2y^2+y^4)$.
29. $(x^2+y^2)(x^4-x^2y^2+y^4)$.
30. $(x^2+y^2)(x+y)(x-y)$.
31. $(x-y)(x^2+xy+y^2)(x^3+y^3)$.
32. $(x+y)(x^2-xy+y^2)(x^3-y^3)$.
33. $(x^3-y^3)(x^3+y^3)$.
34. $(x^2+2\,x+1)(x^2-2\,x+1)$.
35. $(x+y)^2(x+y)^2$.
36. $(x+y)^3(x+y)^2$.
37. $(a^2+2\,ab+b^2)(a-b)^2$.
38. $(3\,ax^2-7)(3\,ax^2+7)$.
39. $(3\,ax^2-y)(3\,ax^2-y)$.
40. $(x^4-x^2y^2)(x^4+x^2y^2)$.
41. $(9\,a^2+3\,ab+b^2)(3\,a-b)$.
42. $(x+y+z)(x-y-z)$.
43. $(a+b-c+d)^2$.
44. $(x-y-z-v)^2$.

QUOTIENT OF TWO POWERS OF THE SAME BASE

76. Illustrative Problem. To divide x^6 by x^4.

Since by § 50 the quotient times the divisor equals the dividend, we seek an expression which multiplied by x^4 equals x^6.

Since by Principle XV two powers of the same base are multiplied by adding their exponents, the expression sought must be that power of x whose exponent added to 4 equals 6. Hence the exponent of the quotient is $6-4=2$. That is, $x^6 \div x^4 = x^{6-4} = x^2$.

ORAL EXERCISES

Perform the following indicated divisions:

1. $2^4 \div 2^2$.
2. $2^3 \div 2^2$.
3. $2^4 \div 2$.
4. $3^3 \div 3^2$.
5. $3^4 \div 3$.
6. $3^4 \div 3^2$.
7. $9^{11} \div 9^{10}$.
8. $5^{13} \div 5^{12}$.
9. $7^{24} \div 7^{22}$.
10. $8^3 \div 8$.
11. $6^4 \div 6^2$.
12. $a^3 \div a^2$.
13. $a^4 \div a^3$.
14. $m^4 \div m^2$.
15. $x^4 \div x^2$.
16. $t^{14} \div t^4$.
17. $m^3 \div m$.
18. $n^6 \div n^2$.
19. $(20)^4 \div (20)$.
20. $(101)^{14} \div (101)^{13}$
21. $(41)^7 \div (41)^6$.

The preceding exercises illustrate

Principle XVI

77. **Rule.** *The quotient of two powers of the same base is a power of that base whose exponent is the exponent of the dividend minus that of the divisor.*

For the present only those cases are considered in which the exponent of the dividend is greater than or equal to that of the divisor.

Notice that Principle XVI does not apply to powers of *different* bases.

E.g. The result of $3^4 \div 2^2$ does not equal any integral base to the power, $4 - 2$. This division can be performed only by first multiplying out both dividend and divisor. Thus, $3^4 \div 2^2 = 81 \div 4 = 20\frac{1}{4}$.

ORAL EXERCISES

Perform the following indicated divisions by means of Principle XVI:

1. $2^7 \div 2^3$.
2. $a^7 \div a^3$.
3. $3^4 \div 3^2$.
4. $x^4 \div x^2$.
5. $3^{3n} \div 3^{2n}$.
6. $x^{4n} \div x^{2n}$.
7. $3^{2a-1} \div 3^{a-2}$.
8. $5^{n+5} \div 5^{n+2}$.
9. $x^{a+4} \div x^{a+2}$.
10. $t^{4a} \div t^a$.
11. $x^{2a+b} \div x^{a+b}$.
12. $w^{2x} \div w^x$.
13. $(17)^{14} \div (17)^{13}$.
14. $5x^6 \div 2x^2$.
15. $(12)^4 \div (12)^3$.

WRITTEN EXERCISES

In the following, use Principles III, V, and XVI:

1. $(3 \cdot 2^4 + 5 \cdot 2^3) \div 2^2$.
2. $(3 \cdot 4^5 - 5 \cdot 4^4) \div 4^4$.
3. $(a^3b - a^4b^2) \div a^2$.
4. $(4x^3 + 3x^4) \div x^2$.
5. $(a^2m^4 - b^2m^3) \div m^3$.
6. $(4x^2 - 5x^3 + x^4) \div x^2$.
7. $(3a^3 + 9a^4 - 2a^5) \div a^3$.
8. $(12x^3y - 11x^2y^2 + 5x^4) \div x^2$.
9. $(8a^3 + 12a^4) \div 4a^3$.
10. $(9x^5 + 15x^6) \div 3x^2$.
11. $(8x^4 - 16x^8) \div 8x^4$.
12. $(15x^7 - 10x^3) \div 5x^2$.
13. $(4ax^2 - 8a^2x) \div 4x$.
14. $(8a^2x^3 - 12ax^2) \div 2x^2$.
15. $(12a^3x^2 - 8x^3a^2) \div 4a^2$.
16. $(16a^3x^3 - 8x^2a^3) \div 2x^2$.
17. $(24x^2y^3 + 16x^3y^2) \div 8x^2$.
18. $(24x^3y^2 - 16x^2y^3) \div 8y^2$.
19. $(12x^3y - 16x^2y^2) \div 4y$.
20. $(x^a + x^{3a}) \div x^a$.
21. $(x^m + x^{4m}) \div x^m$.
22. $(x^{3m} - x^{2m}) \div x^m$.
23. $(y^{3a} - y^{2a}) \div y^a$.
24. $(a^{2m} - a^{3m}) \div a^m$.
25. $(4x^8 - 8x^7 + 16x^6 - 20x^5 + 12x^4) \div 4x^4$.
26. $(4x^{8n} - 8x^{7n} + 16x^{6n} - 20x^{5n} + 12x^{4n}) \div 4x^{4n}$.
27. $(3xy^4 - 6x^2y^3 + 12x^3y^2 - 24y^5) \div 3y^2$.
28. $(3xy^{4a} - 6x^2y^{3a} + 12x^3y^{2a} - 24y^{5a}) \div 3y^{2a}$.
29. $(a^7b^2 - a^6b^3 + a^5b^2 - 4a^4) \div a^4$.
30. $(a^{7k}b^2 - a^{6k}b^3 + a^{5k}b^2 - 4a^{4k}) \div a^{4k}$.
31. $(x^{10}y^5 - x^9y^6 + x^8y^7 - x^7y^8 + x^6y^9) \div x^6$.
32. $(x^{10a}y^5 - x^{9a}y^6 + x^{8a}y^7 - x^{7a}y^8 + x^{6a}y^9) \div x^{6a}$.

78. **Division by Monomials.** In dealing with the quotient of two monomials the indicated division may be written in the form of a fraction and the factors common to dividend and divisor may be cancelled, that is, divided out of both numerator and denominator, just as in arithmetic.

Example 1. $15\,a^3b^2c \div 3\,a^2bx^2y = \dfrac{15\,a^3b^2c}{3\,a^2bx^2y} = \dfrac{5\,abc}{x^2y}$.

Example 2. $12\,a^2x \div 3\,ax = \dfrac{12\,a^2x}{3\,ax} = \dfrac{4\,a}{1} = 4\,a$.

ORAL EXERCISES

Give the following quotients in their simplest forms:

1. $\frac{2\,a^2b^3}{ab^2}$.
2. $\frac{4\,x^3y^4}{2\,xy^3}$.
3. $\frac{6\,m^2n^3}{3\,mn}$.
4. $\frac{8\cdot 15\cdot 14}{4\cdot 5\cdot 7}$.
5. $\frac{4\,x^6y^4}{x^3y^2}$.
6. $\frac{5\,x^2y^3}{2\,x^2y^3}$.
7. $\frac{8\,a^4b^4}{4\,a^2b^3}$.
8. $\frac{9\,x^7y^4}{3\,x^6y^2}$.
9. $\frac{a^4b^6c^2}{a^2b^2c^2}$.
10. $\frac{6\cdot 8\cdot 10}{3\cdot 4\cdot 2}$.
11. $\frac{12\,x^2y^3}{6\,x^2y}$.
12. $\frac{10\,a^7b^7c^2}{5\,a^7b^6c}$.
13. $\frac{48\,x^4y^{n+4}}{12\,x^3y^{n+3}}$.
14. $\frac{24\,a^{2n}b^{3n}}{6\,a^nb^{2n}}$.
15. $\frac{84\,x^{2m+3}y^{n+4}}{21\,x^{2m+1}y^{n+1}}$.
16. $\frac{16\,a^{2n}b^{3n}}{4\,a^nb^{2n}}$.

WRITTEN EXERCISES

Write each in the fractional form and cancel.

Divide:

1. $4\cdot 7\cdot 9\,x^5$ by $2\cdot 3\,x^2$.
2. $12\cdot 8\cdot 20\,x^8$ by $2\cdot 4\cdot 5\,x^5$.
3. $6\,x^3y^2z$ by $2\,xyz$.
4. $6^4\cdot 5^4\cdot x^3$ by $6^2\cdot 5^2\cdot x^2$.
5. $12\,x^{12}y^{13}$ by $4\,xy^3z$.
6. $5\,a^4b^{11}c$ by ab^4c^2.
7. $10\,x^4b^{14}c^3$ by $2\,xb^4c$.
8. $36\,x^4y^3$ by $6\,x^3y^5$.
9. $4\,x^2y^3 - 3\,x^3y^2$ by x^2y^2.
10. $18\,x^4y^4 - 12\,x^3y^3 + 6\,x^2y^2$ by $6\,x^2y^2$.
11. $49\,a^4 + 21\,a^3 - 7\,a$ by $7\,a$.
12. $12\,ax^4y^3 - 16\,a^2x^3y^2 + 8\,a^3xy$ by $4\,axy$.
13. $2\,x^{3a} + 4\,x^{4a} - 8\,x^{2a}$ by $2\,x^a$.
14. $6\,x^{2n+1} + 12\,x^{3n+1} - 10\,x^{n+1}$ by $2\,x^{n+1}$.
15. $4\,x^{13} - 6\,x^{11}b - 10\,x^4c$ by $2\,x^4$.
16. $10\,a^3b^2 - 20\,a^2b^3 + 15\,a^4b^4$ by $5\,a^2b^2$.

79. **Negative Exponents.** The process of division by subtracting exponents leads in certain cases to interesting results.

Thus, $x^4 \div x^4 = x^{4-4} = x^0$, which seems to have no meaning, since an exponent has been defined only when it is a *positive integer*. The exponent zero cannot indicate, as in the case of a positive integer, how many times the base is used as a factor.

We know, however, that $x^4 \div x^4 = 1$, since any number divided by itself equals unity. Thus, any number with the exponent zero is equal to unity. Hence if we use the symbol x^0, it must be interpreted to mean 1, *no matter what number is represented by x.*

Again by this process, $x^2 \div x^4 = x^{2-4} = x^{-2}$, which seems to have no meaning, since negative exponents have not been defined. But we know that if we divide numerator and denominator by x^2, as in arithmetic, $x^2 \div x^4 = \frac{x^2}{x^4} = \frac{1}{x^2}$

Hence if we use the symbol x^{-2}, it must be interpreted to mean $\frac{1}{x^2}$. Similarly $x^{-3} = \frac{1}{x^3}$, $x^{-6} = \frac{1}{x^6}$, etc. Negative exponents and also fractional exponents are the subjects of more advanced work and are considered in detail in the Intermediate Course.

ORAL EXERCISES

Express the following quotients by means of negative exponents after first cancelling any common factors, as in § 78.

1. $\frac{1}{x^2}$.
2. $\frac{1}{x^3}$.
3. $\frac{1}{x^4}$.
4. $\frac{1}{a^5}$.
5. $\frac{x}{x^3}$.
6. $\frac{a}{a^4}$.
7. $\frac{b}{b^3}$.
8. $\frac{c}{c^4}$.
9. $\frac{ab}{a^3b^3}$
10. $\frac{a^2x}{a^3x^2}$.
11. $\frac{a^3x^2}{a^5x^3}$.
12. $\frac{a^4y^4}{a^5y^5}$.

ORAL EXERCISES

Express the following in the fractional form:

1. a^{-2}.
2. b^{-3}.
3. x^{-1}.
4. a^{-4}.
5. x^{-6}.
6. a^{-10}.
7. $a^{-2}y^{-3}$.
8. $x^{-1}y^{-2}$.
9. xy^{-4}.
10. $a^{-3}b^{2}$.
11. $a^{-4}b^{-3}$.
12. $x^{-2}y^{-3}$.

WRITTEN EXERCISES

By means of negative exponents write each of the following without using the fractional form:

1. $\frac{5\,ax^3}{x^2y^3}$.
2. $\frac{a^2b^3}{x^5y^4}$.
3. $\frac{4\,xy^3}{x^3y^6}$.
4. $\frac{2\,ab^2}{x^3y^9}$.
5. $\frac{4}{x^my^n}$.
6. $\frac{4\,a}{x^{2m}y^{n+4}}$.
7. $\frac{4\,b^2c}{b^4c^6x^2}$.
8. $\frac{3\,xy}{x^ay^b}$.
9. $\frac{2}{x^{m+4}y^{5a+3}}$.

Express the following without using negative exponents:

10. $ax^{-3}y^{-2}$.
11. $a^{-3}yx^{-2}$.
12. $a^{-2}y^{-3}x^{2}$.
13. $xy^{-a}z^{-1}$.
14. $x^{-2}yz^{-2a}$.
15. $x^{-a}y^{-b}z$.
16. $x^{-3a}y^{-2b}z^{-1}$.
17. $a^{-1}b^{-2}c^{-a}$.
18. $a^{-m}b^{-2n-3}c$.

HISTORICAL NOTE

Exponents. The expression "power" is used by Alkarismi to denote the *square* of a number. Up to the time of Vieta it was customary to use different letters (if letters were used at all) to express the square, the cube, etc. of a number. Thus, if R represented a number, Q might represent the square of it and C the cube of it. Vieta wrote A, A quad, A cube, etc., for A, A^2, A^3, etc. Harriot wrote aa for a^2, aaa for a^3, etc. Descartes (1637) established the usage of the forms a^2, a^3, etc.

John Wallis (1616–1703) explained the meaning of negative and of fractional exponents. Thus he wrote x^{-1} for $\frac{1}{x}$, x^{-2} for $\frac{1}{x^2}$, etc.

Sir Isaac Newton (1642–1727) used exponents of any magnitude. Thus he used not only x^3, x^{-3}, $x^{\frac{1}{3}}$, but also such expressions as $x^{\sqrt{2}}$. These latter need not be discussed here.

DIVISION BY A POLYNOMIAL

80. Illustrative Example. Consider the product

$$(x^2 + 2xy + y^2)(x + y) = x^2(x + y) + 2xy(x + y) + y^2(x + y).$$

The products $x^2(x + y)$, $2xy(x + y)$, and $y^2(x + y)$ are called **partial products**, and their sum, $x^3 + 3x^2y + 3xy^2 + y^3$, the **complete product**.

In dividing $x^3 + 3x^2y + 3xy^2 + y^3$ by $x + y$, the quotient must be a polynomial such that when its terms are multiplied by $x + y$ the results are these partial products.

The work may be arranged as follows:

Dividend or product =	$x^3 + 3x^2y + 3xy^2 + y^3$	$x + y$, divisor.
1st product, $x^2(x+y)$ =	$x^3 + x^2y$	$x^2 + 2xy + y^2$,
Dividend minus 1st product =	$2x^2y + 3xy^2 + y^3$	[quotient.
2d product, $2xy(x + y)$ =	$2x^2y + 2xy^2$	
Dividend minus 1st and 2d products =	$xy^2 + y^3$	
3d product, $y^2(x + y)$ =	$xy^2 + y^3$	
Dividend minus 1st, 2d, and 3d products =	0	

Explanation. Dividing the first term, x^3, of the dividend by the first term, x, of the divisor the quotient is x^2. Multiplying this term of the quotient by the divisor, we obtain the first partial product, $x^3 + x^2y$.

Subtracting the first partial product from the whole product, $x^3 + 3x^2y + 3xy^2 + y^3$, the remainder is $2x^2y + 3xy^2 + y^3$. Dividing the first term, $2x^2y$, of this remainder by x the quotient is $2xy$. The product of $2xy$ and $x + y$ is the second partial product.

In like manner the third partial product is $xy^2 + y^3$.

After subtracting the third partial product the remainder is zero.

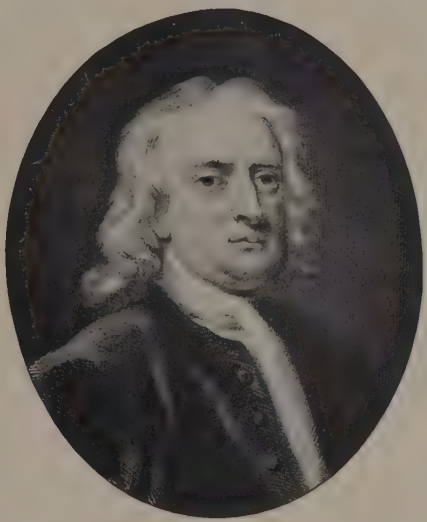
Checking Problems in Division. Problems in division may be checked by substituting any convenient values for the letters. For instance, in the above example, if $x=1$, $y = 1$, we have:

Divisor $= x + y = 1 + 1 = 2$.
Quotient $= x^2 + 2xy + y^2 = 1 + 2 + 1 = 4$.
Dividend $= x^3 + 3x^2y + 3xy^2 + y^3 = 1 + 3 + 3 + 1 = 8$.
We know that we should have

$$Dividend = Divisor \times Quotient.$$

But $8 = 2 \times 4$. Hence the correctness of the division is shown.

But since division by zero is impossible, care must be taken in checking not to select such values for the letters as will reduce the divisor to zero.



Sir Isaac Newton (1642–1727), probably the greatest mathematician of all time, was born near Grantham in Lincolnshire, England, in the same year that Galileo died.

He entered Cambridge in 1661 and began at once a career of unequalled productive study. The binomial theorem was one of his early discoveries. Later Newton laid the foundation of the calculus.

His chief work, the *Principia*, published in 1687, aimed "to apply mathematics to the phenomena of nature." This work placed him in the very front rank for all time among mathematicians, physicists, and astronomers.

Example. Divide $x^3 - 7x^2 + 2x^4 - 1 + 5x$ by $2x - 1 + x^2$.

Solution. We first arrange both dividend and divisor according to the descending powers of x. [divisor.

Dividend or product =	$2x^4 + x^3 - 7x^2 + 5x - 1$	x^2+2x-1,
1st product, $2x^2(x^2+2x-1) =$	$2x^4 + 4x^3 - 2x^2$	$2x^2-3x+1$,
Dividend minus 1st product =	$-3x^3 - 5x^2 + 5x - 1$	[quotient.
2d product, $-3x(x^2+2x-1) =$	$-3x^3 - 6x^2 + 3x$	
Dividend minus 1st and 2d products =	$+x^2 + 2x - 1$	
3d product, $1 \cdot (x^2+2x-1) =$	$x^2 + 2x - 1$	
Dividend minus 1st, 2d, and 3d products =	0	

Check. Substituting $x = 2$, we get $21 \div 7 = 3$.

81 From a consideration of the preceding examples the process of dividing by a polynomial is described as follows:

1. *Arrange the terms of dividend and divisor according to descending (or ascending) powers of some common letter. As the division proceeds, arrange each remainder in the same way.*

2. *Divide the first term of the dividend by the first term of the divisor. This quotient is the first term of the quotient.*

3. *Multiply the first term of the quotient by the divisor and subtract the product from the dividend.*

4. *Divide the first term of this remainder by the first term of the divisor, obtaining the second term of the quotient. Multiply the divisor by the second term of the quotient and subtract, obtaining a second remainder.*

5. *Continue in this manner until the last remainder is zero, or until a remainder is found whose first term does not contain as a factor the first term of the divisor.*

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ORAL EXERCISES

Arrange each of the following in descending powers of the letter involved:

1. $16x - 26x^2 + 3 + 11x^3 + x^4$.
2. $8a - 5a^2 + 8 - 4a^5 + 2a^4 - a^3$.
3. $3y^3 + 4 + 6y^4 - 2y - y^2$.
4. $4z^2 - 18 + 6z^4 + 18z$.

WRITTEN EXERCISES

Divide the following. Check the results in Examples 1 to 13, being careful to substitute such numbers for the letters as do not make the divisor zero:

1. $a^2+2ab+b^2$ by $a+b$.
2. $a^2-2ab+b^2$ by $a-b$.
3. $a^3-3a^2b+3ab^2-b^3$ by $a-b$.
4. $2x^3+2x^2y-4x^2-x-4xy-y$ by $x+y$.
5. $x^3-x^2y+xy^2-y^3$ by $x-y$.
6. x^3+4x^2+x-6 by $x+3$.
7. x^3+x+4x^2-6 by $x-1$.
8. $x^4-6x^3+2x^2-3x+6$ by $x-1$.
9. $x^3+3xy^2+3x^2y+y^3$ by x^2+y^2+2xy.
10. x^3-8x^2+75 by $x-5$.
11. $2a^3+19a^2b+9ab^2$ by $2a+b$.
12. x^3+y^3 by $x+y$.
13. x^3-y^3 by $x-y$.

Divide the following:

14. $x^4+x^3y+xy^3+y^4$ by $x+y$.
15. $x^4+y^4+x^2y^2$ by x^2-xy+y^2.
16. x^4-y^4 by $x-y$.
17. $x^4+x^3-12x^2+14x-4$ by x^2-3x+2.
18. $2x^4+11x^3-26x^2+16x-3$ by x^2+7x-3.
19. $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$ by $x^2+2xy+y^2$
20. $x^5+10x^2-x^4-27x^3-30x-200$ by $x^2-4x-10$.
21. $3x^2-4xy+8xz-4y^2+8yz-3z^2$ by $x-2y+3z$.
22. $9r^2s^2-4r^2t^2+4rst^2-s^2t^2$ by $3rs-2rt+st$.
23. $9a^2b^2+16x^2-4a^2-36b^2x^2$ by $3ab+6bx-2a-4x$.
24. $x^3+x^2y+x^2z-xyz-y^2z-yz^2$ by x^2-yz.
25. $a^5-a^3b^2+a^4b+a^2b+ab^2-b^3$ by a^2+ab-b^2.

82. **Division not Exact.** In case the division is not exact, the remainder may be placed over the divisor in the fractional form as in arithmetic.

Example. Divide $20\,a^2 - 4 + 18\,a^4 + 18\,a - 19\,a^3$ by $2\,a^2 - 3\,a + 4$.

Solution. Arranging dividend and divisor according to the descending powers of a, we have

		[divisor.
Dividend or product:	$18\,a^4 - 19\,a^3 + 20\,a^2 + 18\,a - 4$	$2\,a^2 - 3\,a + 4$,
1st product:	$18\,a^4 - 27\,a^3 + 36\,a^2$	$9\,a^2 + 4\,a - 2$,
Dividend minus 1st product:	$8\,a^3 - 16\,a^2 + 18\,a - 4$	[quotient.
2d product:	$8\,a^3 - 12\,a^2 + 16\,a$	
Dividend minus 1st and 2d products:	$-4\,a^2 + 2\,a - 4$	
3d product:	$-4\,a^2 + 6\,a - 8$	
Dividend minus all products:	$-4\,a + 4$	

Since $2\,a^2$ is not contained in $-4\,a$, the division ends and $-4\,a + 4$ is the remainder. As in arithmetic we write this as the numerator of a fraction whose denominator is the divisor. Hence, the complete result is

$$9\,a^2 + 4\,a - 2 + \frac{4 - 4\,a}{2\,a^2 - 3\,a + 4}.$$

WRITTEN EXERCISES

In the following express the remainders in the fractional form:

Divide:

1. $4\,x^4 - 2\,x^2 + 6\,x^3 + 4\,x - 7$ by $2\,x^2 - x + 2$.
2. $8\,a^5 - 4 + 7\,a^4 + 2\,a^3 - 2\,a^2$ by $a^2 - 3\,a + 1$.
3. $3 - 4\,x + 7\,x^2 - 3\,x^3 + x^4$ by $x - 4$.
4. $2\,x^3 - 6 + 2\,x^2 - 6\,x + 6\,x^4$ by $x^2 + 5$.
5. $13\,x - 7\,x^2 + 8\,x^4 + 3\,x^3 + 14$ by $x^2 + 3$.
6. $21 - 7\,x - 9\,x^3 + 8\,x^2 + x^4$ by $x^3 + 7$.
7. $12\,x^3 - 4\,x^2 + 8\,x + 16\,x^4$ by $x^2 + x + 4$.
8. $35\,a - 14 + 21\,a^2 + 7\,a^3$ by $a - 5$.
9. $16\,x^7 - 14\,x^5 + 2\,x^3 - x + 2$ by $2\,x^4 - 4\,x^2 + 8$.
10. $2\,x^4 - 11\,x^3 + 26\,x^2 + 18\,x + 3$ by $x^2 - 7\,x + 3$.
11. $6\,x^5 - 2\,x^4 + 2\,x^3 - x^2 + x - 6$ by $3\,x^2 - 4\,x + 2$.

83. Division by Detached Coefficients. When both dividend and divisor are arranged in descending powers of the same letter, the work of dividing by a polynomial may be shortened by omitting the letters and writing only the coefficients.

Example. Divide $2x^4 + x^3 - 7x^2 + 5x - 1$ by $x^2 + 2x - 1$.

Writing coefficients only,

$$\begin{array}{r|l} 2+1-7+5-1 & 1+2-1 \\ \underline{2+4-2} & \underline{2-3+1} \\ -3-5+5-1 & \\ \underline{-3-6+3} & \\ +1+2-1 & \\ \underline{+1+2-1} & \end{array}$$

Since dividend and divisor are arranged in descending powers of x, we know that the quotient will be so arranged also. Since the first term of the quotient contains $x^4 \div x^2$, the quotient starts with x^2 and is then

$$2x^2 - 3x + 1.$$

This example is worked without detached coefficients on page 103.

If any terms are lacking in the dividend or divisor, zero must be written as the coefficient of each such term, as in the following example.

Example. Divide $x^4 + x^2 + 1$ by $x^2 - x + 1$.

Since the third and first powers are lacking in the dividend, we write zero in place of each.

$$\begin{array}{r|l} 1+0+1+0+1 & 1-1+1 \\ \underline{1-1+1} & \underline{1+1+1} \\ +1+0+0+1 & \\ \underline{+1-1+1} & \\ +1-1+1 & \\ \underline{+1-1+1} & \end{array}$$

Hence, the quotient starts with $x^4 \div x^2 = x^2$, and is, then, $x^2 + x + 1$.

WRITTEN EXERCISES

Divide by detached coefficients Examples 3, 6, 7, 8, 10, 12, 13, 16, and 20, page 104.

NOTE. — Division by detached coefficients is much used in certain topics in higher algebra where the work is still further abbreviated by a process called *synthetic division*.

REVIEW QUESTIONS

1. Make a diagram to show how to multiply $(7+4)$ by $(11+8)$ without first uniting the terms of the binomials. Multiply $(a+b)$ by $(c+d)$ in the same manner.

Multiply $(12-3)$ by $(9-7)$ in two ways and compare results. State the principle by which two polynomials are multiplied.

2. Describe a convenient manner of arranging the work in multiplying polynomials. What kind of terms in the product are placed in the same column? Find the product of $7x-3y+1$ and $2x-8y-3$, arranging your work this way.

3. State Principle XIII. How do you arrange the factors in multiplying $3a^2b$, $2ab^2$, and $5a^4b^3$?

4. Define a positive integral exponent. Explain the difference between an exponent and a coefficient.

5. Under what circumstances are exponents added in multiplication? State Principle XV. Use this principle to show that $(a^2)^3=a^6$, $(a^3)^4=a^{12}$.

6. Under what circumstances are exponents subtracted in division? State Principle XVI.

7. What is meant by x^{-2}? How may $\frac{a}{b^3}$ be written with a negative exponent?

8. How may division by a polynomial be shortened by use of detached coefficients?

Add Principles XIII, XIV, XV to your list expressed in symbols:

XIII $\begin{cases} a\cdot b\cdot c = a\cdot c\cdot b = b\cdot c\cdot a, \text{ etc.} \\ a\cdot b\cdot c = ab\cdot c = a\cdot bc \end{cases}$

XIV $a^m\cdot a^n = a^{m+n}$

XV $a^m \div a^n = a^{m-n}$

CHAPTER VI

SPECIAL PRODUCTS AND QUOTIENTS

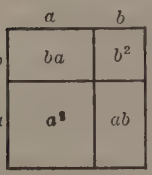
There are certain products and quotients the formulas for which should be memorized. Some of these are collected in this chapter.

THE SQUARE OF A BINOMIAL

84. **The square of the sum of two numbers** is found by ordinary multiplication as in § 67.

E.g. $(a+b)(a+b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$.

Hence, $$(a+b)^2 = a^2 + 2ab + b^2.$$



This product is illustrated in the accompanying figure.

Translated into words, this identity is:

The square of the sum of two numbers is equal to the square of the first, plus twice the product of the two numbers, plus the square of the second.

ORAL EXERCISES

Read at sight the squares indicated by the following:

1. $(x+y)^2$.	6. $(a+4)^2$.	11. $(4+x)^2$.
2. $(m+n)^2$.	7. $(2a+1)^2$.	12. $(1+3x)^2$.
3. $(x+1)^2$.	8. $(2x+1)^2$.	13. $(2+3a)^2$.
4. $(x+2)^2$.	9. $(3+a)^2$.	14. $(2a+3)^2$.
5. $(y+3)^2$.	10. $(3a+1)^2$.	15. $(3a+4)^2$.

By the above formula we may square any binomial sum.

E.g. $(3x+2y)^2 = (3x)^2 + 2 \cdot (3x)(2y) + (2y)^2 = 9x^2 + 12xy + 4y^2$

WRITTEN EXERCISES

Write the squares indicated by the following:

1. $(2a+3b)^2$.	6. $(4c+5a)^2$.	11. $(12+7a)^2$.
2. $(3y+8)^2$.	7. $(8a+2b)^2$.	12. $(10+3x)^2$.
3. $(5x+4)^2$.	8. $(4m+3n)^2$.	13. $(16a+b)^2$.
4. $(3a+7c)^2$.	9. $(6x+5y)^2$.	14. $(8x+3y)^2$.
5. $(2a+9b)^2$.	10. $(x+12)^2$.	15. $(12a+7b)^2$.

85. Similarly, we obtain the **square of the difference of two numbers:**

$$(a-b)^2 = a^2 - 2ab + b^2.$$

That is, *the square of the difference of two numbers is equal to the square of the first, minus twice the product of the two numbers, plus the square of the second.*

Example. By means of this formula, find the square of $a-3b$.

Solution. $(a-3b)^2 = a^2 - 2\cdot a(3b) + (3b)^2 = a^2 - 6ab + 9b^2$.

ORAL EXERCISES

Read at sight the squares indicated by the following:

1. $(x-y)^2$.	5. $(x-3)^2$.	9. $(2-x)^2$.
2. $(m-n)^2$.	6. $(a-4)^2$.	10. $(3-x)^2$.
3. $(x-1)^2$.	7. $(a-5)^2$.	11. $(4-x)^2$.
4. $(x-2)^2$.	8. $(2a-1)^2$.	12. $(1-2a)^2$.

WRITTEN EXERCISES

Write the squares indicated by the following:

1. $(4x-3y)^2$.	6. $(4c-5a)^2$.	11. $(12-7a)^2$.
2. $(5y-3x)^2$.	7. $(8a-3)^2$.	12. $(7x-9y)^2$.
3. $(2a-3b)^2$.	8. $(7x-4y)^2$.	13. $(8a-3b)^2$.
4. $(5x-3)^2$.	9. $(3x-7y)^2$.	14. $(12a-13)^2$.
5. $(4-3x)^2$.	10. $(ab-4c)^2$.	15. $(9a-7b)^2$.

86. Squaring Polynomials. By means of the formulas in §§ 84 and 85, we may square any polynomial.

1. $(a+b+c)^2=[(a+b)+c]^2=(a+b)^2+2(a+b)c+c^2$
$=a^2+2\,ab+b^2+2\,ac+2\,bc+c^2.$

2. $(a+r-s+t)^2=[(a+r)-(s-t)]^2$
$=(a+r)^2-2(a+r)(s-t)+(s-t)^2$
$=a^2+2\,ar+r^2-2\,(as-at+rs-rt)+s^2-2\,st+t^2$
$=a^2+2\,ar+r^2-2\,as+2\,at-2\,rs+2\,rt+s^2-2\,st+t^2.$

WRITTEN EXERCISES

1. $(a+b-c)^2$.
2. $(a-b+c)^2$.
3. $(a-b-c)^2$.
4. $(a-b+3)^2$.
5. $(3\,a-2\,b+5)^2$.
6. $[7\,x-(4r-s)]^2$.
7. $[(a-3)-2(b+c)]^2$.
8. $[(m+3)-(n+a)]^2$.
9. $(a-b+c-d)^2$.
10. $(x-y+z-3)^2$.
11. $(a-x+b-c)^2$.
12. $(2-x+y+z)^2$.

PRODUCT OF THE SUM AND DIFFERENCE OF TWO NUMBERS

87. Examples. Find the products:

$$(x+5)(x-5) \text{ and } (x+a)(x-a).$$

Solutions

$$\begin{array}{r} x+5 \\ x-5 \\ \hline x^2+5x \\ \quad -5x-25 \\ \hline x^2 \qquad -25 \end{array} \qquad \begin{array}{r} x+a \\ x-a \\ \hline x^2+ax \\ \quad -ax-a^2 \\ \hline x^2 \qquad -a^2 \end{array}$$

In each of these examples one factor is the sum of two numbers and the other factor is the difference of the same numbers. In each case the product is the difference of the squares of the numbers. This is expressed by the formula

$$(x+a)(x-a)=x^2-a^2.$$

That is, *the product of the sum and difference of two numbers is equal to the difference of their squares.*

ORAL EXERCISES

Read the following products:

1. $(a+1)(a-1)$.
2. $(a+3)(a-3)$.
3. $(k-b)(k+b)$.
4. $(3-x)(3+x)$.
5. $(2a+3b)(2a-3b)$.
6. $(a+2b)(a-2b)$.
7. $(2b-1)(2b+1)$.
8. $(1+3x)(1-3x)$.
9. $(1-7y)(1+7y)$.
10. $(a-4b)(a+4b)$.
11. $(6a-3b)(6a+3b)$.
12. $(7-9a)(7+9a)$.
13. $(2c+1)(2c-1)$.
14. $(3a+b)(3a-b)$.
15. $(5k+3h)(5k-3h)$.
16. $(9m+3n)(9m-3n)$.

By means of the above formula a product may be written at once whenever the factors can be expressed as the sum and difference of the same two number expressions.

E.g. $(x+y-z)(x+y+z) = [(x+y)-z][(x+y)+z]$
$= (x+y)^2 - z^2$
$= x^2 + 2xy + y^2 - z^2$.

WRITTEN EXERCISES

In this manner form the following products. Verify the first five by formal multiplication.

1. $(4a+5b)(4a-5b)$.
2. $(5-6t^2)(5+6t^2)$.
3. $(3x-2y)(3x+2y)$.
4. $(x^3-y^3)(x^3+y^3)$.
5. $(3x^4-5y^4)(3x^4+5y^4)$.
6. $(16a^2b^3-3c)(16a^2b^3+3c)$.
7. $(24x+12y)(24x-12y)$.
8. $[x+(y-z)][x-(y-z)]$.
9. $(x^m+y^n)(x^m-y^n)$.
10. $[c-(a-b)][c+(a-b)]$.
11. $[x-(y+z)][x+(y+z)]$.
12. $(a+b+c)(a-b-c)$.
13. $(a+b-c)(a-b+c)$.
14. $(a-b+c)(a-b-c)$.
15. $(r-y-z)(r-y+z)$.
16. $(a+b+c)(a+b-c)$.
17. $(x+2y+z)(x+2y-z)$.
18. $(x-2y+z)(x+2y-z)$.
19. $(x-2y-z)(x+2y+z)$.
20. $(a+2b-c)(a-2b+c)$.
21. $(a-2b-c)(a-2b+c)$.
22. $(2a+3b-4c)(2a-3b+4c)$.

BINOMIALS WITH FIRST TERMS ALIKE

88. **Examples.** Find the products:

$(x+2)(x+3)$ $\qquad$ $(x+4)(x-7)$

$(x+5)(x-2)$ $\qquad$ $(x-5)(x-3)$

Solutions

$$\begin{array}{r} x+2 \\ \underline{x+3} \\ x^2+2x \\ \underline{\quad 3x+6} \\ x^2+5x+6 \end{array} \qquad \begin{array}{r} x+4 \\ \underline{x-7} \\ x+4x \\ \underline{-7x-28} \\ x^2-3x-28 \end{array}$$

In like manner, $(x+5)(x-2) = x^2+3x-10$,

and $(x-5)(x-3) = x^2-8x+15$.

From a study of these examples, we deduce the formula

$$(x+a)(x+b) = x^2+(a+b)x+ab.$$

That is, *the product of two binomials having the first terms alike is equal to the square of the first term, plus the first term multiplied by the algebraic sum of the last terms, plus the product of the last terms.*

ORAL EXERCISES

Read the products of the following:

1. $(x+1)(x+2)$.
2. $(x+1)(x+3)$.
3. $(x+1)(x+4)$.
4. $(x+1)(x+5)$.
5. $(x-1)(x+2)$.
6. $(x-1)(x+3)$.
7. $(x-1)(x+4)$.
8. $(x-1)(x+5)$.
9. $(x+2)(x+3)$.
10. $(x-2)(x+3)$.
11. $(x+2)(x-4)$.
12. $(x-1)(x-2)$.
13. $(x-2)(x-3)$.
14. $(x+2)(x-3)$.
15. $(x+3)(x+4)$.
16. $(x-3)(x+4)$.
17. $(x+3)(x-4)$.
18. $(x-3)(x-4)$.
19. $(x+4)(x+5)$.
20. $(x-4)(x+5)$.
21. $(x+4)(x-5)$.
22. $(x-4)(x-5)$.

WRITTEN EXERCISES

Find the following products:

1. $(x+7)(x+3)$.
2. $(x+9)(x+6)$.
3. $(y+6)(y-2)$.
4. $(y-8)(y+3)$.
5. $(c-4)(c-2)$.
6. $(a-8)(a+10)$.
7. $(a+7)(a+6)$.
8. $(a-7)(a+6)$.
9. $(ab+3)(ab+7)$.
10. $(ab-5)(ab-3)$.
11. $(2a+1)(2a+3)$.
12. $(2a-1)(2a-3)$.
13. $(4x+3)(4x+5)$.
14. $(4x-3)(4x+5)$.
15. $(4x+3)(4x-5)$.
16. $(4x-3)(4x-5)$.
17. $(5y+6)(5y+4)$.
18. $(5y+6)(5y-4)$.
19. $(5y-6)(5y+4)$.
20. $(5y-6)(5y-4)$.
21. $(ab+6)(ab+7)$.
22. $(ab-3)(ab+7)$.

In the formula $(x+a)(x+b)=x^2+(a+b)x+ab$, replace a and b by the following values and simplify the results:

23. $a=5, b=3$.
24. $a=8, b=-7$.
25. $a=6, b=-11$.
26. $a=-5, b=-7$.

27. Find the square of 42 by writing it as a binomial, $40+2$.

28. Square the following numbers by writing each as a binomial sum: 51, 53. 93, 91, 102, 202, 301.

29. Find the square of 29 by writing it as a binomial, $30-1$.

30. Square the following numbers by first writing each as a binomial difference: 28, 38, 89, 77, 99, 198, 499, 998, 999.

31. Find the product of 41 and 39, first indicating the product thus, $(40+1)(40-1)$.

32. Find the following products by writing each pair of factors as the sum and difference of two numbers:

(1) $62 \cdot 58$. (3) $53 \cdot 47$. (5) $17 \cdot 13$.
(2) $27 \cdot 33$. (4) $102 \cdot 98$. (6) $99 \cdot 101$.

THE SQUARE OF A TRINOMIAL

89. **Example.** By multiplication find the square of $a+b+c$, and reduce the result to simplest form.

Solution. By means of the formula in § 84 we can perform this multiplication by a short method; namely,

$$\begin{aligned}(a+b+c)^2 &= [(a+b)+c]^2 \\ &= (a+b)^2 + 2(a+b)c + c^2 \\ &= a^2 + 2ab + b^2 + 2ac + 2bc + c^2.\end{aligned}$$

How many terms are there in the product? How many are squares? How many are of the type $2ab$?

From this we get the following rule:

The square of a trinomial consists of the sum of the squares of its terms plus twice the product of each term by each succeeding term.

In symbols this is

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

The above rule may be used to find the square of $a-b+c$ as follows:

$$\begin{aligned}a-b+c &= a+(-b)+c = a^2+(-b)^2+c^2+2a(-b)+2ac+2(-b)c \\ &= a^2+b^2+c^2-2ab+2ac-2bc.\end{aligned}$$

Similarly, $(2a+b-3c)^2$

$$\begin{aligned}&= (2a)^2 + b^2 + (-3c)^2 + 2(2a)b + 2(2a)(-3c) + 2b(-3c) \\ &= 4a^2 + b^2 + 9c^2 + 4ab - 12ac - 6bc.\end{aligned}$$

Hence, *In the square of a trinomial, the squared terms are all positive and the double products are negative when one factor is negative, otherwise they are positive.*

ORAL EXERCISES

Give the squares of the following:

1. $x+y+z$.	5. $a+b+2$.	9. $a-2b+3$.
2. $x+y-z$.	6. $a+b-2$.	10. $a+2b-3$.
3. $x-y-z$.	7. $a-b-2$.	11. $a-2b-3$.
4. $x-y+z$.	8. $a-b+2$.	12. $a+2b+3$.

ORAL EXERCISES

Give the squares of the following:

1. $3-y-z$.
2. $3-x+y$.
3. $3+a+b$.
4. $a-3+b$.
5. $a+3-b$.
6. $a-3-b$.
7. $-x+y+z$.
8. $-x-y+z$.
9. $-x-y-z$.
10. $-a+b+2$.
11. $-a-b+2$.
12. $-a+b-2$.
13. $x+y+1$.
14. $x+y-1$.
15. $x-y-1$.
16. $x-y+1$.
17. $-x-y+1$.
18. $-x+y-1$.

WRITTEN EXERCISES

Find the squares of the following:

1. $3a-2b+c$.
2. $3a+2b-c$.
3. $4a-3b+2c$.
4. $4a-3b-2c$.
5. $2a+4b-6c$.
6. $x-4y+3z$.
7. $3x^2+2xy+3y^2$.
8. $a^2-b^2+c^2$.
9. $2a^2+3b^2+c^2$.
10. $ab+ac+bc$.
11. $2a^2-3b^2+4c^2$.
12. $a^2b^2+b^2c^2+c^2a^2$.
13. $2a^2b+ab^2+ab$.
14. $ab+a^2b^2+a^3b^3$.
15. $2x^2+x+1$.
16. $3x^3+2x^2+x$.
17. $ax+by+cz$.
18. ax^3+bx^2+cx.
19. $a^2b^2-b^2c^2-a^2c^2$.
20. $a^2x^2-x^2y^2+y^2z^2$.
21. $a^2x+ay+x^2y$.
22. $3a-4x+3b$.
23. $5a-6b+7c$.
24. $3x-5y-9z$.

25. Find the square of $a+b+c+d$ by writing it in the form $[(a+b+c)+d]$. Study this product and make a rule for squaring a polynomial of four terms.

Find the squares of the following:

26. $x+y+z+w$.
27. $a-b+c-d$.
28. $a-2b-c-2d$.
29. $m+2n+3r+s$.
30. $2a-3b+c-d$.
31. $3x+2y-z+w$.

CUBE OF A BINOMIAL

90. Example. Find the cube of $a+b$ by first finding the square of $a+b$ and then multiplying this result by $a+b$. How many terms are there in the product?

From this we get the following formula:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

In words this is:

The cube of a binomial is equal to the cube of the first term, plus three times the square of the first term multiplied by the second, plus three times the first term multiplied by the square of the second, plus the cube of the second term.

The above rule may be applied to find the cube of $a-b$, thus

$$(a-b)^3 = [a+(-b)]^3 = a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3$$
$$= a^3 - 3a^2b + 3ab^2 - b^3.$$

Which terms in the product are negative and why?

Translate into words the formula:

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

EXERCISES

Find the following cubes. Read the first six at sight.

1. $(x+y)^3$.
2. $(x-y)^3$.
3. $(c+d)^3$.
4. $(c-d)^3$.
5. $(m+n)^3$.
6. $(m-n)^3$.
7. $(2x+3y)^3 = (2x)^3 + 3(2x)^2 \cdot 3y + 3 \cdot 2x(3y)^2 + (3y)^3$.
8. $(2x-3y)^3$.
9. $(x+2y)^3$.
10. $(x-3y)^3$.
11. $(3a-b)^3$.
12. $(3a+2b)^3$.
13. $(3a-2b)^3$.
14. $(x-1)^3$.
15. $(x+1)^3$.
16. $(2x+a)^3$.
17. $(2x-a)^3$.
18. $(b+6)^3$.
19. $(2a-5b)^3$.

Historical Note. The Hindus and Arabs had the formulas for $(a+b)^2$ and $(a+b)^3$. Of course their formulas were not stated in terms of our symbols. Vieta used the formula for $(a+b)^4$.

QUOTIENTS DERIVED FROM SPECIAL PRODUCTS

91. From the special products given in §§ 84–88, we may at once derive certain special quotients by observing that if either factor be used as a divisor of the product, the other factor is the quotient.

Thus, from

$$(a+b)(a+b)=a^2+2ab+b^2, \quad (1)$$
$$(a-b)(a-b)=a^2-2ab+b^2, \quad (2)$$
$$(a+b)(a-b)=a^2-b^2, \quad (3)$$
$$(x+a)(x+b)=x^2+(a+b)x+ab, \quad (4)$$

we derive

$$(a^2+2ab+b^2)\div(a+b)=a+b, \quad (1)$$
$$(a^2-2ab+b^2)\div(a-b)=a-b, \quad (2)$$
$$\left.\begin{aligned}(a^2-b^2)\div(a-b)&=a+b\\(a^2-b^2)\div(a+b)&=a-b\end{aligned}\right\}, \quad (3)$$
$$\left.\begin{aligned}[x^2+(a+b)x+ab]\div(x+a)&=x+b\\[x^2+(a+b)x+ab]\div(x+b)&=x+a\end{aligned}\right\}. \quad (4)$$

EXERCISES

Give the following quotients orally if possible:

Divide:

1. $x^2+2xy+y^2$ by $x+y$.
2. $x^2+4xy+4y^2$ by $x+2y$.
3. $x^2-6xy+9y^2$ by $x-3y$.
4. x^2-y^2 by $x-y$.
5. x^4-y^4 by x^2+y^2.
6. $25a^2-16b^2$ by $5a-4b$.
7. $1-9c^2$ by $1-3c$.
8. $9c^2-1$ by $3c+1$.
9. $(a+b)^2-c^2$ by $a+b-c$
10. $(a+b)^2-c^2$ by $a+b+c$.
11. $x^2-(y+z)^2$ by $x-y-z$.
12. $x^2-(y+z)^2$ by $x+y+z$.
13. x^2+5x+6 by $x+3$.
14. x^2-5x+6 by $x-3$.
15. $a^2+7a+12$ by $a+4$.
16. $a^2+7a+12$ by $a+3$.
17. $a^2-7a+12$ by $a-3$.
18. c^2+c-12 by $c+4$.
19. c^2+c-12 by $c-3$.
20. $x^2-2x-15$ by $x-5$.

DIVIDING THE SUM OR DIFFERENCE OF TWO CUBES

92. Examples.

$$
\begin{array}{r|l}
a+b & a^3+b^3 \mid a^2-ab+b^2 \\
\cline{2-2}
 & a^3+a^2b \\
 & -a^2b \\
 & -a^2b-ab^2 \\
\cline{2-2}
 & ab^2+b^3 \\
 & ab^2+b^3 \\
\cline{2-2}
\end{array}
\qquad\qquad
\begin{array}{r|l}
a-b & a^3-b^3 \mid a^2+ab+b^2 \\
\cline{2-2}
 & a^3-a^2b \\
\cline{2-2}
 & a^2b \\
 & a^2b-ab^2 \\
\cline{2-2}
 & ab^2-b^3 \\
 & ab^2-b^3 \\
\cline{2-2}
\end{array}
$$

From these examples we have the two formulas:

$$(a^3+b^3) \div (a+b) = a^2-ab+b^2$$
$$(a^3-b^3) \div (a-b) = a^2+ab+b^2$$

That is,

The sum of the cubes of two numbers is divisible by the sum of the numbers, and the quotient is the sum of the squares of the numbers ***minus*** *their product.*

The difference of the cubes of two numbers is divisible by the difference of the numbers, and the quotient is the sum of the squares of the numbers ***plus*** *their product.*

WRITTEN EXERCISES

After performing the division as indicated, review these examples, giving the quotients orally.

Divide:

1. x^3+y^3 by $x+y$.
2. x^3-y^3 by $x-y$.
3. 2^3+3^3 by $2+3$.
4. r^3+s^3 by $r+s$.
5. $1+a^3$ by $1+a$.
6. $1-b^3$ by $1-b$.
7. a^3+1 by $a+1$.
8. a^3-1 by $a-1$.
9. $a^3+8\,b^3$ by $a+2\,b$.
10. $a^3-8\,b^3$ by $a-2\,b$.
11. $x^3-27\,y^3$ by $x-3\,y$.
12. $1+x^6$ by $1+x^2$.
13. $8+a^9$ by $2+a^3$.
14. $1+125\,x^3$ by $1+5\,x$.
15. 2^3x^3-1 by $2\,x-1$.
16. $64\,a^3-27\,b^3$ by $4\,a-3\,b$.
17. $1-8\,m^3$ by $1-2\,m$.
18. $1-2^3m^6$ by $1-2\,m^2$

USING SPECIAL PRODUCTS IN EQUATIONS

Ability to form rapidly the special products given by the formulas in §§ 84–88 is important in solving certain kinds of equations. See page 86.

Example. Solve $(x+1)^2+(x-2)^2=2(x+3)(x-5)+31$. (1)

Solution. Forming the special products, we have

$$x^2+2x+1+x^2-4x+4=2x^2-4x-30+31. \quad (2)$$

Transposing and collecting terms,

$$2x=-4, \quad (3)$$

$$x=-2. \quad (4)$$

WRITTEN EXERCISES

Solve the following equations:

1. $x^2=(x-3)(x+6)-12.$
2. $(1-x)(1-3x)-(1+x)^2=2x^2-18.$
3. $(a+1)(a+2)+a(a+3)=2a(a+5)+2.$
4. $(b+3)(b-2)-(b+1)^2=-2(b+4)-1.$
5. $(z-5)^2+(z-4)(z-1)=2z(z-3)-25.$
6. $(a+4)^2+(a-1)(2a+5)=(a+4)(3a+2).$
7. $(a-1)(3a-1)-(a+1)^2=2a^2-18.$
8. $(6-a)^2+(a-3)(2a-5)=(3a+1)(a-3)+84.$
9. $(7a-18)(a+4)-(a-1)^2=6(a+2)^2-79.$
10. $(2b-30)(b-1)-5b^2=6b-3(b+5)^2+65.$
11. $(5-c)^2+(7-c)^2+(9-c)^2=(c-1)(3c-58)-93.$
12. $(5c-3)(2+c)-4(c-1)^2=(c+1)^2+54.$
13. $(8-4c)(5-c)=(c+1)^2+(c+3)(3c-8)+218.$
14. $(y-1)^2+4(y+1)^2+(1-y)(5y+6)=15y-29.$
15. $x(x+3)+(x+1)(x+2)=2x(x+5)+2.$
16. $x^2=(x-3)(x+7)-19.$
17. $(5+5x)(3-x)+2(x+1)^2+3(x+1)(x-7)=17(x+1).$
18. $(8+3x)(4-x)+(x-1)(x-2)+2(x+5)^2=105.$

PROBLEMS

1. Find two consecutive integers whose squares differ by 51.

2. Find two consecutive integers whose squares differ by 97.

3. Find two consecutive integers whose squares differ by a. Show from the form of the equation obtained that a must be an *odd* integer.

4. There is a square field such that if each of its dimensions is increased by 5 rods, its area is increased 625 square rods. How large is the field?

Suggestion. If a side of the original field is w, then its area is w^2, and the area of the enlarged field is $(w + 5)^2$.

5. A rectangle is 9 feet longer than it is wide. A square whose side is 3 feet longer than the width of the rectangle is equal to the rectangle in area. What are the dimensions of the rectangle?

6. A rectangle is 16 feet longer than it is wide. If its length is increased by 4 feet and its width is decreased by 3 feet, its area is decreased by 50 square feet. What are its dimensions?

7. A field is 20 rods longer than it is wide. If its length is increased by 8 rods and its width is decreased by 5 rods, the area is decreased by 50 square rods. What are the dimensions of the field?

8. A farmer has a plan for a granary which is to be 15 feet longer than wide. He finds that if the length is decreased 3 feet and the width is increased 2 feet, the floor space will be increased by 16 square feet. What are the dimensions?

9. If the length of a rectangular flower bed is decreased 4 feet and its width is decreased 1 foot, its area will be decreased by 39 square feet. What are its present dimensions, if its length is 8 feet greater than its width?

MISCELLANEOUS EXERCISES

Perform the indicated operations:

1. $(2a-3b+c)^2$.
2. $(5x-y-4z)^2$.
3. $(3m-4n+2p)^2$.
4. $(2a-3b)^3$.
5. $(ab-cd)^3$.
6. $(4x-5y)^3$.
7. $[(a^2+3)-(b^2-2)]^2$.
8. $[(x^2-2)+(y^2-3)]^2$.
9. $(x^2-7x+6)\div(x-6)$.
10. $(144m^4-64n^6)\div(12m^2-8n^3)$.
11. $[(x-y)^2-z^2]\div(x-y+z)$.
12. $[(x+y)^2-z^2]\div(x+y-z)$.
13. $(125x^3-27y^3)\div(5x-3y)$.

REVIEW QUESTIONS

State in words each of the following formulas:

$$(a+b)^2=a^2+2ab+b^2.$$

$$(a-b)^2=a^2-2ab+b^2.$$

$$(a+b)(a-b)=a^2-b^2.$$

$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$$

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3.$$

$$(a^2-b^2)\div(a+b)=a-b.$$

$$(a^2-b^2)\div(a-b)=a+b.$$

$$(a^3+b^3)\div(a+b)=a^2-ab+b^2.$$

$$(a^3-b^3)\div(a-b)=a^2+ab+b^2.$$

$$\left.\begin{aligned}[x^2+(a+b)x+ab]\div(x+a)&=x+b\\ [x^2+(a+b)x+ab]\div(x+b)&=x+a\end{aligned}\right\}.$$

Jan. 13, 1925

Second Semester

CHAPTER VII

FACTORS OF ALGEBRAIC EXPRESSIONS

93. **Factors in Arithmetic and Algebra.** Factors are of great importance in arithmetic. Thus from the multiplication table, we know the factors of such numbers as 25, 42, 49, 54, 63, etc. Likewise in algebra the factors of certain algebraic expressions are so important that they must be known at sight.

94. **Prime Expressions.** An algebraic expression is said to be **prime** if it is divisible only by itself and 1. Factors of an expression, when they cannot themselves be further factored, are called *prime factors.*

Thus, 2, 3, x, $x+2$, a^2+b^2, are prime expressions.

Case I: MONOMIAL FACTORS

$$ax + ay + az = a(x + y + z).$$

95. **Common Monomial Factors.** If the terms of a polynomial contain a common monomial factor, the polynomial may be divided by the monomial, and the quotient and the divisor are factors of the polynomial.

E.g.
$$a^2 - ab = a(a - b);$$
$$5xy - 3x^2y + 4x^3y = xy(5 - 3x + 4x^2),$$
$$x^4 + 3x^3y + 3x^2y^2 + xy^3 = x(x^3 + 3x^2y + 3xy^2 + y^3).$$

Observe that factoring the various parts of a polynomial does not factor the polynomial.

E.g. $a^2 + ax + ab + by$ is not factored by writing it

$$a(a + x) + b(a + y).$$

Removing a common factor from the terms of a polynomial is nothing more than the application of Principle I.

ORAL EXERCISES

Read the following and give the factors of each:

1. $x^3 + x^2 + x$.
2. $4a^2 + 3a + a^3$.
3. $a^3b^2 + a^2b^3$.
4. $3xy - 4x^2y^2$.
5. $3ab^2c - 2a^2bc^2$.
6. $5x^3y^4 - 10x^4y^3$.
7. $2m^3n - 3mn^3$.
8. $4ax^2y - 6a^2xy^2$.
9. $4xy^2 - 2xy$.
10. $6xy - 3x^2y$.
11. $4x^3y^4 + 8x^7y^6$.
12. $8x^7y^6 - 12x^5y^8$.
13. $x^2y + xy^2$.
14. $xy^3 - x^2y^5$.
15. $x^5 + 4x^4 + 5x^3 + 3x^2$.
16. $8a^3b^2 - 12a^2b^3$.
17. $3a^2b^3c^2 - 9a^4b^3c^2$.
18. $5xyz + 25x^2y^2z^2$.
19. $4ab - 12a^2b$.
20. $3a^2b^2 - 9a^3b^2$.
21. $5ab^2 - 10ab^4$.
22. $6a^3b^3 - 3ab$.
23. $7a^2b^3 + 14ab$.
24. $8a^3b + 24a^4b$.
25. $9x^2y^3 - 6x^4y$.
26. $10x^5y^2 - 5x^2y^5$.

WRITTEN EXERCISES

Factor the following polynomials:

1. $7a^4b^3 + 14a^2b + 21ab^3$.
2. $13a^4b - 16a^3b^4 - 2a^3b^2$.
3. $15xy^4 - 20x^3y + x^2y^2$.
4. $9v^3w^4 + 21v^4w^3 - 18v^2w^2$.
5. $12a^4b^3 - 8a^3b^4 - 6a^2b^2$.
6. $11a^4x^2 - 44a^3x^4 + 33ax$.
7. $72ab^3x^4 - 36a^2b^2x^4 - 48a^4b^3x^3$.
8. $84b^7x^9y^4 + 18b^5x^3y^4 + 12b^3x^2y^2$.
9. $17a^4b^3c^2 + 51a^3b^4c^3 - 34a^2b^2c^4$.
10. $38a^{12}b^{14}c^4 - 76a^{11}b^{12}c^3 - 76a^{13}b^{11}c^2$.
11. $4x^{2a}y^{3b} + 6x^{3a}y^{2b} - 8x^{5a}y^{4b}$.
12. $3a^{4n}b^{3n} + 6a^{6n}b^{3n} - 12a^{5n}b^{5n}$.
13. $2x^ay^b + 4x^{3a}y^{4b} - 6x^{4a}y^{3b}$.
14. $3x^{2a}y^{2b} + 6x^{3a}y^{2b} - 9x^{4a}y^{4b}$.
15. $4x^{2a}y^b + 6x^{3a}y^{2b} - 9x^{2a}y^{3b}$.
16. $5x^{2a}y^b + 15x^{5a}y^{2b} - 20x^{6a}y^{3b}$.

Case II: TRINOMIAL SQUARES

$$\begin{cases} a^2 + 2ab + b^2 = (a+b)(a+b), & (1) \\ a^2 - 2ab + b^2 = (a-b)(a-b). & (2) \end{cases}$$

96. **Square Root.** If an expression is the product of two equal factors, either of these factors is called its *square root.*

Thus, a is the square root of a^2, since $a^2 = a \cdot a$.

In §§ 84 and 85 we found by multiplication:

$$(a+b)(a+b) = a^2 + 2ab + b^2,$$

and $$(a-b)(a-b) = a^2 - 2ab + b^2.$$

Hence, $a+b$ is the square root of $a^2 + 2ab + b^2$ and $a-b$ is the square root of $a^2 - 2ab + b^2$.

97. **Trinomial Squares.** A trinomial which is the square of a binomial is called a *trinomial square.*

Thus $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ are trinomial squares.

98. From a study of the two trinomials $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$, we learn to distinguish whether *any* given trinomial is a perfect square, as in the following examples :

1. $x^2 + 4x + 4$ is in the form of (1), since x^2 and 4 are squares each with the sign +, and $4x$ is twice the product of the square roots of x^2 and 4. Hence

$$x^2 + 4x + 4 = x^2 + 2(2x) + 2^2 = (x+2)(x+2) = (x+2)^2.$$

2. $x^2 - 4x + 4$ is in the form of (2), since it differs from (1) only in the sign of the middle term. Thus

$$x^2 - 4x + 4 = x^2 - 2(2x) + 2^2 = (x-2)(x-2) = (x-2)^2.$$

99. The foregoing examples lead to the following

Rule. *A trinomial is a perfect square if it contains two terms which are squares, each with the sign +, while the third term, whose sign is either + or −, is twice the product of the square roots of the other two.*

The square root of such a trinomial is the sum or the difference of these square roots according as the sign of the third term is + or −.

ORAL EXERCISES

Determine whether the following are trinomial squares, and if so indicate the two equal factors.

1. $a^2 + 2\,ad + d^2$.
2. $x^2 + 2\,xy + y^2$.
3. $x^2 - 2\,xy + y^2$.
4. $x^4 + 2\,x^2y^2 + y^4$.
5. $x^4 - 2x^2y^2 + y^4$.
6. $m^2 + n^2 - 2\,mn$.
7. $r^2 + s^2 + 2\,rs$.
8. $4\,x^2 - 8\,xy + 4\,y^2$.
9. $a^6 + b^6 + 2\,a^3b^3$.
10. $a^8 + b^8 - 2\,a^4b^4$.
11. $64 + t^2 - 16\,t$.
12. $16 + x^2 - 8\,x$.
13. $9 - 6\,y + y^2$.
14. $25\,x^2 + 16\,y^2 + 40\,xy$.
15. $4\,m^2 + n^2 + 2\,mn$.
16. $100 + s^2 + 20\,s$.
17. $64 + 49 + 112$.
18. $16\,a^2 + 25\,b^2 - 50\,ab$.

WRITTEN EXERCISES

Decide which of the following are trinomial squares. Find the square roots of all such:

1. $9 + 2 \cdot 3 \cdot 4 + 16$.
2. $x^2 + 4\,y^2 + 4\,xy$.
3. $9\,x^2 + 18\,xy + 9\,y^2$.
4. $4\,x^2 + 4\,xy + y^2$.
5. $4\,x^2 + 8\,xy + 4\,y^2$.
6. $25\,x^2 + 12\,xy + 4\,y^2$.
7. $16\,x^2 + 16\,xy + 4\,y^4$.
8. $9\,r^2 + 36\,rs + 25\,s^2$.
9. $16\,x^6 + 8\,x^3y + y^2$.
10. $4\,x^8 + 12\,a^3x^4 + 9\,a^6$.
11. $a^{10} + 6\,a^5b + 9\,b^2$.
12. $(a+1)^2 + 2(a+1)b + b^2$.
13. $(x+3)^2 + 4(x+3)y + 4\,y^2$.
14. $x^6 + 12\,x^3 + 36$.
15. $a^4 + 18\,a^2 + 12$.
16. $121 + 4\,x^3 - 44\,x^4$.
17. $16\,x^4 + 64\,y^4 - 64\,x^2y^2$.
18. $81\,a^2 - 216\,a + 144$.
19. $4\,a^2 + 8\,ab^2 + 4\,b^2$.
20. $9\,b^4 + 18\,b^2c^4 + 9\,c^8$.
21. $4\,x^2 + 4\,y^2 - 8\,xy$.
22. $9\,a^2 - 16\,ab + 4\,b^2$.
23. $9\,a^4 - 24\,a^2b + 16\,b^2$.
24. $25 + 49\,x^2 - 70\,x$.
25. $-30\,ab^2 + 9\,a^2 + 25\,b^4$.
26. $16\,a^2 - 24\,ab + 9\,b^2$.
27. $36\,x^2 - 84\,x + 49$.
28. $25 - 90 + 81$.
29. $64\,x^2 - 32\,x + 9$.
30. $(3 + a)^2 + b^2 - 2\,b(3 + a)$.

Case III: THE DIFFERENCE OF TWO SQUARES

$$a^2 - b^2 = (a + b)(a - b).$$

100. In § 87, we found by multiplication,

$$(a + b)(a - b) = a^2 - b^2.$$

Hence we have the formula

$$a^2 - b^2 = (a + b)(a - b)$$

From this formula we obtain the following

Rule. *Every binomial which is the difference between two perfect squares is the product of two binomial factors; namely, the sum and the difference of the square roots of these squares.*

E.g. $16\,x^2 - 9\,y^2$ is the difference of the two squares, $(4\,x)^2$ and $(3\,y)^2$. Hence we have

$$16\,x^2 - 9\,y^2 = (4\,x)^2 - (3\,y)^2 = (4\,x + 3\,y)(4\,x - 3\,y).$$

ORAL EXERCISES

In each of the following expressions determine whether it is the difference of two squares, and if so, find the factors.

1. $x^2 - 4\,y^2$.
2. $9\,x^2 - 36\,y^2$.
3. $x^4 - b^2$.
4. $4\,x^2 - 9\,b^2$.
5. $16\,a^4 - 9\,b^4$.
6. $64 - b^2$.
7. $1 - b^2$.
8. $a^2 - 1$.
9. $1 - 9\,x^4$.
10. $4 - 36\,a^2$.
11. $1 - 64\,a^8$.
12. $144\,x^2b^4 - 1$.
13. $36\,a^4b^6 - c^2$.
14. $1 - (x + y)^2$.
15. $4 - (x + 2\,y)^2$.
16. $16\,a^2 - 25\,b^2$.
17. $49\,x^2 - 4\,y^2$.
18. $225 - 64\,x^2y^4$.
19. $81\,a^2 - 144\,y^2$.
20. $5^8 - 3^8$.
21. $x^4 - 81\,y^2$.
22. $a^2 - (x + y)^2$.
23. $(x + y)^2 - a^2$.
24. $(x - y)^2 - a^2$.
25. $a^2 - (x - y)^2$.
26. $(a + 3)^2 - 16$.
27. $(x + 3)^2 - 25$.
28. $25 - (a + b)^2$.
29. $36 - 4(a + b)^2$.
30. $9(a - b)^2 - 4$.
31. $16(a - b)^2 - 4\,c^2$.
32. $16(a - b)^2 - 9\,d^2$.
33. $4 - 9(a + b)^2$.
34. $9 - 16(a + b)^2$.
35. $9\,a^2 - 4(x + y)^2$.
36. $25(x - y)^2 - 4\,z^2$.
37. $36(x + y)^2 - 25\,z^2$.
38. $64\,x^2 - 49(a + b)^2$.
39. $81\,a^4 - 36(b + c)^2$.

101. Following is another example of an expression which may be written as the difference of two squares:

E.g. $a^2 + b^2 + 2ab - c^2 = (a + b)^2 - c^2 = (a + b + c)(a + b - c)$.

WRITTEN EXERCISES

Factor each of the following:

1. $x^2 - (y - z)^2$.
2. $(x - y)^2 - z^2$.
3. $a^2 + b^2 - 2ab - 4$.
4. $x^2 + y^2 - 2xy - z^2$.
5. $4a^2b^2 - (a^2 + b^2 - c^2)^2$.
6. $a^2 - (b^2 + c^2 + 2bc)$.
7. $(2a - 5)^2 - (3a + 1)^2$.
8. $(3x^2 - y)^2 - (x + y)^2$.
9. $(3a - 2b)^2 - (8a + 5b)^2$.
10. $(3m - 4)^2 - (2m + 3)^2$.
11. $(2r + s)^2 - (3r - s)^2$.
12. $81 - (a + b + c)^2$.
13. $x^4 + 2x^2 + 1 - 4a^2$.
14. $a^2 - (x + 2y)^2$.
15. $9x^2 - (a - b)^2$.
16. $25m^2 - (3r + 2s)^2$.

102. Expressions Reducible to the Difference of Two Squares.*

Example. The trinomial $a^4 + a^2b^2 + b^4$ would become a perfect square if its middle term were $2a^2b^2$ instead of a^2b^2. Hence, if we add a^2b^2 to this term and subtract a^2b^2 from the whole expression, we shall have the difference of two squares.

Thus, $a^4 + a^2b^2 + b^4 = a^4 + 2a^2b^2 + b^4 - a^2b^2 = (a^2 + b^2)^2 - a^2b^2$.

WRITTEN EXERCISES

Factor the following:

1. $x^4 + x^2y^2 + y^4$.
2. $x^8 + x^4y^4 + y^8$.
3. $a^4 + 4b^4 = (a^4 + 4a^2b^2 + 4b^4) - 4a^2b^2$.
4. $m^8 + 4n^8$.
5. $a^4 + a^2 + 1$.
6. $y^8 + y^4 + 1$.
7. $4x^4 + 11x^2y^2 + 9y^4$.
8. $x^4 - 14x^2y^2 + 25y^4$.
9. $4a^4 - 29a^2b^2 + 25b^4$.
10. $16a^4 + 20a^2b^2 + 9b^4$.
11. $a^{4n} + a^{2n}b^{2n} + b^{4n}$.
12. $x^4 - 12x^2y^2 + 4y^4$.
13. $a^4 - 17a^2b^2 + 16b^4$.

* This article may be omitted without destroying the continuity.

Case IV: THE SUM OF TWO CUBES

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

103. In § 92 we found

$$(a^3 + b^3) \div (a + b) = a^2 - ab + b^2.$$

Since Dividend = Divisor × Quotient, we have

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

From this formula we obtain the following

Rule. *The sum of the cubes of two numbers is the product of two factors, one of which is the sum of the numbers, and the other is the sum of their squares minus their product.*

E.g. (1) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

(2) $8a^6 + 27b^3 = (2a)^3 + (3b)^3$.
$= (2a + 3b)(4a^2 - 2a \cdot 3b + 9b^2)$.

(3) $x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$.

Notice the difference between the trinomial $x^2 - xy + y^2$ and the trinomial square $x^2 - 2xy + y^2$.

EXERCISES

Determine whether each of the following is the sum of two cubes, and if so find the factors. Read 1–6 at sight.

1. $x^3 + y^3$.	10. $8a^3 + 27b^3$.	19. $64x^2 + 27y^3$.
2. $a^3 + 8b^3$.	11. $8a^3 + 64b^3$.	20. $8^3 + 10^3$.
3. $27a^3 + b^3$.	12. $a^3x^6 + x^9a^3$.	21. $1 + 729x^6$.
4. $8a^3 + 1$.	13. $1 + 8a^3b^3$.	22. $x^6 + y^{12}$.
5. $1 + 64x^3$.	14. $64x^3 + 343$.	23. $a^9 + b^3$.
6. $2^3 + 3^3$.	15. $1 + a^3$.	24. $27r^3 + 125s^3$.
7. $125 + 729$.	16. $a^3 + 9b^3$.	25. $x^9 + 27y^6$.
8. $1 + 125x^6$.	17. $125x^3 + y^6$.	26. $64 + a^9$.
9. $27x^6 + 1$.	18. $1 + x^3$.	27. $a^3b^6 + x^3y^9$.

28. Find whether $x^3 + y^3$ is exactly divisible by $x - y$.

Case V: THE DIFFERENCE OF TWO CUBES

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

104. In § 92 we found

$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2.$$

Since Dividend = Divisor × Quotient, we have

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

From this formula we obtain the following

Rule. *The difference of the cubes of two numbers is the product of two factors, one of which is the difference of the numbers, and the other is the sum of their squares plus their product.*

E.g. (1) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

(2) $8a^3 - 64b^3 = (2a)^3 - (4b)^3$
$= (2a - 4b)(4a^2 + 2a \cdot 4b + 16b^2)$.

(3) $a^9 - b^9 = (a^3)^3 - (b^3)^3 = (a^3 - b^3)(a^6 + a^3b^3 + b^6)$.

Notice the difference between the factor $x^2 + xy + y^2$, and the trinomial square $x^2 + 2xy + y^2$.

EXERCISES

Determine whether each of the following is the difference of two cubes, and if so, find the factors. Read 1–6 at sight.

1. $r^3 - s^3$.	8. $1 - 8a^3$.	15. $27x^3 - 64$.
2. $x^3 - 1$.	9. $64x^3 - y^3$.	16. $2^3x^3 - 1$.
3. $1 - x^3$.	10. $27 - 125a^3$.	17. $8x^4 - y^3$.
4. $x^6 - y^6$.	11. $x^3 - y^6$.	18. $64a^3 - 27b^3$.
5. $1 - y^6$.	12. $x^9 - 8$.	19. $1 - 729x^6$.
6. $y^6 - 1$.	13. $1 - 125x^3$.	20. $x^6 - y^{12}$.
7. $1 - a^{10}$.	14. $8 - 27x^3$.	21. $27r^3 - 125s^3$.

22. Also factor Examples 4, 5, 6, 19, and 20 as the difference of two squares; and then resolve these factors still further.

Case VI: TRINOMIALS OF THE FORM x^2+px+q

105. In § 88 were found such products as

$$(1)\ (x+5)(x+2)=x^2+7x+10.$$
$$(2)\ (x-5)(x-2)=x^2-7x+10.$$
$$(3)\ (x+5)(x-2)=x^2+3x-10.$$
$$(4)\ (x-5)(x+2)=x^2-3x-10.$$

All these are included in the form

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

in which the *coefficient of x is the algebraic sum of a and b and the last term is their product.*

106. It is possible to determine at sight whether a trinomial is of the form just considered, and if it is, to find the factors by inspection.

Illustrative Examples. Determine whether the following trinomials can be factored by inspection:

1. $x^2+7x+12$. The question is whether two numbers can be found such that their sum is $+7$ and their product 12. 3 and 4 are such numbers. Hence, $x^2+7x+12=(x+3)(x+4)$.

2. $x^2-5x-14$. Since the product of the numbers sought is -14, one number must have the sign $-$ and the other $+$; and since their sum is -5, the one having the greater absolute value must have the sign $-$. The numbers are -7 and $+2$, and we have $x^2-5x-14=(x-7)(x+2)$.

3. $x^2-7x+12=(x-3)(x-4)$. Since $(-3)(-4)=+12$ and $(-3)+(-4)=-7$.

4. $x^2+4x-12=(x+6)(x-2)$. Since $(+6)(-2)=-12$ and $(+6)+(-2)=+4$.

These examples lead to the following

Rule. *To factor a trinomial such as x^2+px+q, we try to find two numbers, a and b, whose product is q and whose algebraic sum is p. If two such numbers can be found, then the factors are $x+a$ and $x+b$.*

ORAL EXERCISES

In each of the following state what is the product of the two numbers to be found and what is their sum. Then give the factors.

1. x^2+3x+2.
2. x^2-3x+2.
3. x^2-x-2.
4. x^2+x-2.
5. x^2+4x+3.
6. x^2-2x-3.
7. x^2+2x-3.
8. x^2-4x+3.
9. x^2+5x+4.
10. x^2-5x+4.
11. x^2-3x-4.
12. x^2+3x-4.
13. x^2+6x+5.
14. x^2-4x-5.
15. x^2+4x-5.
16. x^2-6x+5.
17. x^2+7x+6.
18. x^2-5x-6.
19. x^2+5x-6.
20. x^2-7x+6.
21. x^2+5x+6.
22. x^2-x-6.
23. x^2+x-6.
24. x^2-5x+6.
25. $x^2+7x+12$.
26. x^2-x-12.

Give orally or in writing the factors of the following:

27. x^2+x-12.
28. $x^2-7x+12$.
29. x^2+6x+8.
30. x^2-2x-8.
31. x^2+2x-8.
32. x^2-6x+8.
33. x^2+8x+7.
34. x^2-8x+7.
35. x^2-6x-7.
36. x^2+6x-7.
37. x^2+6x+9.
38. x^2-6x+9.
39. $x^2+10x+9$.
40. $x^2-10x+9$.
41. x^2+8x-9.
42. x^2-8x-9.
43. $x^2+7x+10$.
44. $x^2-7x+10$.
45. $x^2+3x-10$.
46. $x^2-3x-10$.
47. $x^2+8x+12$.
48. $x^2-8x+12$.
49. $x^2-4x-12$.
50. $x^2+4x-12$.
51. $x^2+9x+20$.
52. $x^2-9x+20$.

107. It is not always possible to factor by inspection expressions of the form $x^2 + px + q$; for it may be that there are no integers whose product is q and whose algebraic sum is p.

E.g. Given $x^2 + 5x + 3$. It is easily seen that there are no two integers such that their sum is $+5$ and their product $+3$.

WRITTEN EXERCISES

Determine whether each of the following trinomials can be factored by inspection, and if so, find the factors.

1. $x^2 + 11x + 24$.
2. $x^2 + 2x - 35$.
3. $x^2 - 3x - 40$.
4. $x^2 - 2x - 24$.
5. $x^2 + x - 30$.
6. $x^2 - 3x - 8$.
7. $x^2 + 2x - 24$.
8. $a^2 - 4a - 32$.
9. $a^2 + 4a - 32$.
10. $b^2 + 15b + 56$.
11. $b^2 + 8b + 15$.
12. $b^2 - b - 56$.
13. $b^2 + b - 56$.
14. $c^2 - 3c - 15$.
15. $x^2 - 15x + 56$.
16. $x^2 + 15x - 54$.
17. $x^2 - 14x - 95$.
18. $y^2 + 21y + 98$.
19. $y^2 - 7y - 98$.
20. $x^2 - 19x + 78$.
21. $x^4 + 18x^2 + 77$.
22. $x^4 - 5x^2 - 104$.
23. $a^2 + 32a + 240$.
24. $a^4 - 11a^2 + 28$.
25. $a^4 - 11a^2 - 60$.
26. $a^2 - 14a - 51$.
27. $a^2 - 3a - 54$.
28. $x^4 - 8x^2 - 32$.
29. $a^6 - 3a^3 - 154$.
30. $x^2 - 10x + 25$.
31. $a^2b^6 - 13ab^3 - 30$.
32. $x^2 - 17xyz + 72y^2z^2$.
33. $r^8 + 6r^4s - 91s^2$.
34. $a^4c^4 + 9a^2c^2 - 162$.
35. $a^2 + 11a - 210$.
36. $m^4 + 4m^2n + 4n^2$.
37. $s^2t^2 - 15st - 54$.
38. $a^2b^2 - 27ab + 26$.
39. $l^2 + 13l + 42$.
40. $x^2y^2 - 11xy - 180$.
41. $9a^2 + 24a + 16$.
42. $81a^2 - 99a + 30$.
43. $g^2 + 26g + 133$.
44. $x^2 + 5xy - 84y^2$.
45. $r^2 + 3r - 154$.
46. $u^2 + 38uv + 165v^2$.

Case VII: TRINOMIALS OF THE FORM ax^2+bx+c.

108. Examples.

(1) $2x+5$ / $3x+2$ / $6x^2+15x$ / $4x+10$ / $6x^2+19x+10$

(2) $2x+5$ / $3x-2$ / $6x^2+15x$ / $-4x-10$ / $6x^2+11x-10$

In Example (1), the products $3x \cdot 2x = 6x^2$ and $2 \cdot 5 = 10$ are called *end products* and $2 \cdot 2x = 4x$ and $5 \cdot 3x = 15x$ are called *cross products.* Likewise, in Example (2), $6x^2$ and -10 are *end products* and $-4x$ and $15x$ are *cross products.*

In each case we see that the *final result is a trinomial, two of whose terms are the end products while the third term is the algebraic sum of the cross products.*

Likewise, examine the following:

(3) $2x-5$ / $3x+2$ / $6x^2-15x$ / $4x-10$ / $6x^2-11x-10$

(4) $2x-5$ / $3x-2$ / $6x^2-15x$ / $-4x+10$ / $6x^2-19x+10$

WRITTEN EXERCISES

In this manner obtain the following products:

1. $(2a+3)(a+3)$.
2. $(4a-1)(3a+2)$.
3. $(2x+5)(x-7)$.
4. $(7r+8)(3r-6)$.
5. $(2x+8)(9x-4)$.
6. $(3m-1)(4m+3)$.
7. $(5s-7)(2s-4)$.
8. $(2x-1)(7x+4)$.
9. $(4n-9)(5n-7)$.
10. $(8y-1)(5y+11)$.
11. $(t-5)(t+4)$.
12. $(5x-y)(2x-3y)$.
13. $(3x-2y)(x+3y)$.
14. $(4a-3y)(a+y)$.
15. $(3r-2s)(2r+s)$.
16. $(5m-n)(2m+n)$.
17. $(5a+3x)(3a-4x)$.
18. $(4a-5b)(a+3b)$.
19. $(3a+5b)(a-b)$.
20. $(3c-7d)(2c+3d)$.
21. $(2a-3b)(3a+2b)$.
22. $(6x-5y)(2x+3y)$.

109. Factoring $ax^2 + bx + c$ by Inspection. Trinomials in the form $ax^2 + bx + c$ may sometimes be factored by inspection.

Example 1. Factor $5x^2 + 16x + 3$.

If this is the product of two binomials they must be such that the end products are $5x^2$ and 3 and the sum of the cross products $16x$.

One pair of binomials having the required end products is $5x + 3$ and $x + 1$. Others are $5x + 1$ and $x + 3$; $5x - 1$ and $x - 3$; and $5x - 3$ and $x - 1$.

It is convenient to write down these possible pairs of factors as follows, as if arranged for multiplication:

$5x + 3$	$5x - 3$	$5x + 1$	$5x - 1$
$x + 1$	$x - 1$	$x + 3$	$x - 3$

The sum of the cross products in the first pair is $8x$, in the second pair $-8x$, in the third pair $16x$, and in the fourth $-16x$. Since $16x$ is the middle term required, the factors are $5x + 1$ and $x + 3$.

Example 2. Factor $6x^2 - 19x + 10$.

Pairs of binomials which give the right end products are

$3x + 5$	$3x - 5$	$2x + 5$	$2x - 5$
$2x + 2$	$2x - 2$	$3x + 2$	$3x - 2$

Of these, the ones which give the right cross products are $2x - 5$ and $3x - 2$.

Hence $6x^2 - 19x + 10 = (2x - 5)(3x - 2)$.

WRITTEN EXERCISES

Factor the following:

1. $2x^2 + 5x + 2$.
2. $2x^2 + 3x - 2$.
3. $2x^2 - 3x - 2$.
4. $2x^2 - 5x + 2$.
5. $2x^2 + 7x + 3$.
6. $2x^2 - 7x + 3$.
7. $2x^2 + 5x - 3$.
8. $2x^2 - 5x - 3$.

From these examples we deduce the following

Rule. *To factor a trinomial of the form $ax^2 + bx + c$, write down the possible pairs of binomials which give the proper end products. Select that pair whose cross products give the proper algebraic sum.*

WRITTEN EXERCISES

In this manner factor the following:

1. $3x^2+5x+2$.
2. $3x^2+x-2$.
3. $3x^2-5x+2$.
4. $3x^2-x-2$.
5. $3x^2+7x+2$.
6. $3x^2-7x+2$.
7. $3x^2+5x-2$.
8. $3x^2-5x-2$.
9. $3x^2+17x+10$.
10. $3x^2-17x+10$.
11. $3x^2+13x-10$.
12. $3x^2-13x-10$.
13. $5x^2+6x+1$.
14. $5x^2-6x+1$.
15. $5x^2+4x-1$.
16. $5x^2-4x-1$.
17. $6x^2+7x+1$.
18. $6x^2-7x+1$.
19. $3x^2+4x+1$.
20. $3x^2-4x+1$.
21. $5x^2-17x-12$.
22. $5x^2+17x-12$.
23. $9a^2+9a+2$.
24. $2x^2+11x+12$.
25. $9x^2+36x+32$.
26. $2x^2-x-28$.

In the following try to find the factors without writing all the pairs which give proper end products.

27. $12s^2+11s+2$.
28. $6t^2+7t-3$.
29. $6x^2-x-2$.
30. $5r^2+18r-8$.
31. $14a^2-39a+10$.
32. $5x^2+26x-24$.
33. $2x^2-5x+2$.
34. $2m^2-m-3$.
35. $7c^2-3c-4$.
36. $5x^4+9x^2-18$.
37. $7a^4+123a^2-54$
38. $6c^2-19c+15$.
39. $3a^2-21a+30$.
40. $6d^2+4d-2$.
41. $20a^2-a-99$.
42. $12c^2+25c+12$.
43. $8+6a-5a^2$.
44. $15-5x-10x^2$.
45. $6-5x-4x^2$.
46. $3h^2-13h+14$.
47. $15r^2-r-2$.
48. $2t^2+11t+5$.
49. $10-5x-15x^2$.
50. $5x^2-33x+18$.

Case VIII: FACTORS FOUND BY GROUPING

$$ax + ay + bx + by = (a + b)(x + y).$$

110. Another method of general application will now be applied to polynomials of four terms.

Example 1. Find the factors of $ax + ay + bx + by$.

By Principle I, the first two terms may be added and also the last two.

Thus, $ax + ay + bx + by = a(x + y) + b(x + y)$.

These two compound terms have a *common factor*, $(x + y)$, and may be added with respect to this factor by Principle I.

Thus, $a(x + y) + b(x + y) = (a + b)(x + y)$.

Hence, $ax + ay + bx + by = (a + b)(x + y)$.

Example 2. Factor $ax - ay - bx + by$.

Combining the first two terms with respect to a and the second two with respect to $-b$, we have,

$$ax - ay - bx + by = a(x - y) - b(x - y).$$

Again combining with respect to the factor $x - y$,

$$ax - ay - bx + by = (a - b)(x - y).$$

The success of this method depends upon the possibility of so grouping and combining the terms as to reveal a *common binomial factor.*

WRITTEN EXERCISES

Factor the following:

1. $ab^2 + ac^2 - db^2 - dc^2$.
2. $6\,ms - 15\,nt + 9\,ns - 10\,mt$.
3. $8\,ax - 10\,ay + 4\,bx - 5\,by$.
4. $2\,a^2 + 3\,ak - 14\,an - 21\,nk$.
5. $ac + bc + ad + bd$.
6. $ax^2 - bx^2 - ay^2 + by^2$.
7. $8\,ac - 20\,ad - 6\,bc + 15\,bd$.
8. $2\,ax - 6\,bx + 3\,by - ay$.
9. $5 + 4\,a - 15\,c - 12\,ac$.
10. $15\,b - 6 - 20\,bc + 8\,c$.
11. $2\,n^2 - cn + 2\,nd - cd$.
12. $5\,ax - 15\,ay - 3\,bx + 9\,by$.
13. $3\,xa - 12\,xc - a + 4\,c$.
14. $3\,xy - 4\,mn + 6\,my - 2\,xn$.
15. $7\,mn + 7\,mr - 2\,n^2 - 2\,nr$.
16. $a - 1 + a^3 - a^2$.
17. $3\,s + 2 + 6\,s^4 + 4\,s^3$.
18. $as^2 - 3\,bst - ast + 3\,bt^2$.
19. $3\,mn + 6\,m^2 - 2\,am - an$.
20. $2\,ar + 2\,as + 2\,br + 2\,bs$.

Case IX: THE SQUARE OF A TRINOMIAL

$$a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc = (a + b + c)^2.$$

111. In § 89 we found

(1) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc.$
(2) $(a + b - c)^2 = a^2 + b^2 + c^2 + 2\,ab - 2\,ac - 2\,bc.$
(3) $(a - b + c)^2 = a^2 + b^2 + c^2 - 2\,ab + 2\,ac - 2\,bc.$
(4) $(a - b - c)^2 = a^2 + b^2 + c^2 - 2\,ab - 2\,ac + 2\,bc.$

A study of these forms enables us to determine whether a polynomial of six terms is a perfect square; namely,

(1) *Three of the terms must be squares each with the sign +.*

(2) *Each of the other three terms must be twice the product of the square roots of two of the square terms.*

(3) *The signs of these products must all be +, or else two of them must be − and one +.*

Example. Find whether the following is a perfect square:

$$4\,x^2 - 12\,xy - 16\,xz + 9\,y^2 + 24\,yz + 16\,z^2$$

Solution. The terms $4\,x^2$, $9\,y^2$, and $16\,z^2$ are all squares, each with the sign +. The square roots of these are $2\,x$ or $-2\,x$, $3\,y$ or $-3\,y$, $4\,z$ or $-4\,z$. By trial we find that $2(2\,x)(-3\,y) = -12\,xy$; $2(2\,x)(-4\,z) = -16\,xz$; and $2(-3\,y)(-4\,z) = 24\,yz$.

Hence the given polynomial is equal to $(2\,x - 3\,y - 4\,z)^2$.

WRITTEN EXERCISES

In each of the following determine whether the polynomial is a perfect square, and if so indicate its square root.

1. $x^2 + y^2 + z^2 - 2\,xy + 2xz - 2\,yz.$
2. $a^2 - 8\,ab + 16\,b^2 - 2\,ac + c^2 + 8\,bc.$
3. $9\,x^2 + 4\,y^2 + z^2 - 12\,xy + 6\,xz - 4\,yz.$
4. $y^4 - 4\,y^2 - 8\,xy^2 + 16\,x + 16\,x^2 + 4.$
5. $a^2 + a^2b^2 - 2\,a^2b + 2\,abc - ab^2c + b^2c^2.$
6. $x^6 - 4\,x^5 + 4\,x^4 + 6\,x^3 - 12\,x^2 + 9.$
7. $x^2 + 16\,x^2y^2 + 289 + 8\,x^2y + 34\,x + 136\,xy.$

Case X: THE REMAINDER THEOREM*

112. It is possible to tell whether a binomial like $x - 2$ will exactly divide a polynomial like $x^2 - 5x + 4$ without actually performing the division.

Examples. (1)
$$\begin{array}{r|l} x^2 - 5x + 4 & x - 2 \\ \underline{x^2 - 2x} & x - 3 \\ -3x + 4 & \\ \underline{-3x + 6} & \\ -2 & \end{array}$$

(2)
$$\begin{array}{r|l} x^2 - 5x + 4 & x - 1 \\ \underline{x^2 - x} & x - 4 \\ -4x + 4 & \\ \underline{-4x + 4} & \\ 0 & \end{array}$$

If we substitute 2 for x in $x^2 - 5x + 4$, we get $4 - 10 + 4 = -2$, which is the remainder in Example (1).

If we put 1 for x in $x^2 - 5x + 4$, we get $1 - 5 + 4 = 0$, which is the remainder in Example (2).

These examples illustrate the **remainder theorem.**

Rule: *If we substitute a number* ***k*** *for* ***x*** *in a polynomial involving* ***x****, the resulting number is the remainder arising from dividing the polynomial by* $x - k$.

To make this more evident, let us divide $x^2 - 5x + 4$ by $x - k$.

$$\begin{array}{r|l} x^2 - 5x + 4 & x - k \\ \underline{x^2 - kx} & x + (k - 5) \\ (k - 5)x + 4 & \\ \underline{(k - 5)x - k^2 + 5k} & \\ k^2 - 5k + 4. \text{ Remainder.} & \end{array}$$

We thus see that this remainder is exactly like the dividend with x replaced by k according to the rule.

The object of this rule is to find remainders which are zero, for then the division is exact, and the **divisor is a factor** of the given polynomial.

Hence this rule is also called the **factor theorem.**

For instance, if we put $x = 2$ in $x^2 - 5x + 4$, we get $4 - 10 + 4 = -2$, which is the remainder, according to the rule, when we divide $x^2 - 5x + 4$ by $x - 2$. Hence $x - 2$ is not a factor of $x^2 - 5x + 4$.

But if we put $x = 1$ in $x^2 - 5x + 4$, the remainder is $1 - 5 + 4 = 0$. Hence $x - 1$ is a factor of $x^2 - 5x + 4$.

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* Articles 112–115 may be omitted without destroying the continuity.

113. Finding Factors by the Remainder Theorem. Since one expression is divisible by another only when the remainder is zero, we use the remainder theorem to find factors as in the following examples:

Example 1. Is $x-1$ a factor of x^2-3x+2?

Solution. Substitute 1 for x in x^2-3x+2 and we have $1-3+2=0$. Hence there is no remainder when we divide x^2-3x+2 by $x-1$. That is, $x-1$ is a factor of x^2-3x+2.

Example 2. Show by the remainder theorem that $x-1$ and $x+2$ are factors of x^2+x-2.

Solution. If we substitute 1 for x in x^2+x-2, we get $1+1-2=0$. Hence there is no remainder when we divide x^2+x-2 by $x-1$. That is, $x-1$ is a factor of x^2+x-2.

To apply this test to $x+2$, we write it in the form $x-(-2)$ and then substitute -2 for x in x^2+x-2 and get $4-2-2=0$.

Hence, $x-(-2)=x+2$ is an exact divisor of x^2+x-2. That is, $x+2$ is a factor of x^2+x-2.

Example 3. Factor x^2+4x-5 by the remainder theorem.

We know that if a binomial exactly divides x^2+4x-5, its last term must be a factor of 5.

Hence the only possible binomial divisors are $x-5$, $x+5$, $x-1$, and $x+1$.

If we substitute 5 for x in x^2+4x-5, we get $25+20-5=40$.

Hence the remainder is not zero when we divide x^2+4x-5 by $x-5$, and therefore $x-5$ is not a factor of x^2+4x-5.

If we substitute -5 for x we get $25-20-5=0$. Hence $x-(-5)=x+5$ is a factor of x^2+4x-5.

Likewise, we find that $x-1$ is a factor and $x+1$ is not.

Hence, $x^2+4x-5=(x+5)(x-1)$.

Example 4. Is $x-2$ a factor of x^3-5x^2+7x-2?

By the remainder theorem the remainder is $8-20+14-2=0$. Hence $x-2$ is a factor. To find the other factor, we divide by $x-2$ and get as the quotient x^2-3x+1.

Example 5. Is $x-1$ a factor of x^7-1?

By the remainder theorem, $x=1$ gives $1^7-1=1-1=0$. Hence, $x-1$ is a factor of x^7-1.

ORAL EXERCISES

1. Is $x-1$ a factor of x^3-1?
2. Is $x-1$ a factor of x^3+1?
3. Is $x+1$ a factor of x^3-1?

Suggestion. Put $x=-1$ in x^3-1.

4. Is $x+1$ a factor of x^3+1?
5. Is $x-1$ a factor of x^5-1? of x^5+1?
6. Is $x+1$ a factor of x^5-1? of x^5+1?
7. Is $x-1$ a factor of x^6-1? of x^6+1?
8. Is $x+1$ a factor of x^6-1? of x^6+1?
9. Is $x-1$ a factor of x^7-1? of x^7+1?
10. Is $x+1$ a factor of x^7-1? of x^7+1?
11. Is $x+1$ a factor of x^8-1? of x^8+1?

WRITTEN EXERCISES

1. Is $x+1$ a factor of x^2+3x+2?
2. Is $x-1$ or $x+1$ a factor of x^2-4x+1?
3. Is $x-1$ a factor of x^3-2x^2+2x-1?
4. Find the factors of $x^3-7x^2+11x-5$.
5. Is $x-2$ a factor of x^3-8? of x^3+8?
6. Is $x-y$ a factor of x^4-y^4? of x^4+y^4? of x^5-y^5? of x^5+y^5? of x^6-y^6? of x^6+y^6?

Suggestion. In each case put y in place of x and see whether the expression is reduced to zero.

7. Is $x+y$ a factor of x^4-y^4? of x^4+y^4? of x^5-y^5? of x^5+y^5? of x^6-y^6? of x^6+y^6?
8. Is $x-1$ a factor of $x^{11}-1$? of $x^{12}-1$?
9. Is $x+1$ a factor of $x^{11}-1$? of $x^{12}-1$?
10. Factor a^3-7a+6.
11. Factor x^3+2x^2-x-2.
12. Factor $a^4-a^3-7a^2+a+6$.

FACTORS OF $x^n + y^n$ AND $x^n - y^n$

114. Applying the Remainder Theorem. By use of the remainder theorem we may find under what conditions $x + y$ and $x - y$ are factors of $x^n + y^n$ and $x^n - y^n$.

1. Is $x + y$ a factor of $x^n + y^n$?

If we substitute $-y$ for x we have $(-y)^n + y^n$. This is zero only when $(-y)^n = -y^n$; that is, when n is an *odd integer.*

E.g. $(-y)^5 = -y^5$ but $(-y)^6 = +y^6$.

Hence $x + y$ is a factor of $x^5 + y^5$, but *not* of $x^6 + y^6$.

2. Is $x + y$ a factor of $x^n - y^n$?

Here we have $(-y)^n - y^n$. This reduces to zero only when $(-y)^n = +y^n$; that is, when n is an *even integer.*

E.g. $(-y)^6 = +y^6$ but $(-y)^7 = -y^7$.

Hence $x + y$ is a factor of $x^6 - y^6$, but *not* of $x^7 - y^7$.

3. Is $x - y$ a factor of $x^n + y^n$?

Since $y^n + y^n$ is never zero, $x - y$ is not a factor of $x^n + y^n$.

E.g. $x - y$ is not a factor of $x^3 + y^3$, nor of $x^4 + y^4$.

4. Is $x - y$ a factor of $x^n - y^n$?

Since $y^n - y^n = 0$, $x - y$ is a factor of $x^n - y^n$, for all integral values of n.

E.g. $x - y$ is a factor of $x^3 - y^3$ and also of $x^4 - y^4$.

Summary. From the foregoing examples, we conclude that:

(1) $x + y$ *is a factor of* $x^n - y^n$ *if* n *is* ***even*** *but not if* n *is* ***odd.***
(2) $x - y$ *is a factor of* $x^n - y^n$ *whether* n *is* ***even*** *or* ***odd.***
(3) $x + y$ *is a factor of* $x^n + y^n$ *if* n *is* ***odd*** *but not if* n *is* ***even.***
(4) $x - y$ *is not a factor of* $x^n + y^n$ *in* ***any*** *case.*

115. Special Case. When n is an *even* integer, it is best to factor $x^n - y^n$ as the *difference of two squares.*

E.g. $x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$
$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$.

Also $x^8 - y^8 = (x^4 - y^4)(x^4 + y^4)$
$= (x - y)(x + y)(x^2 + y^2)(x^4 + y^4)$.

EXERCISES

1. Find one factor of $x^9 - y^9$; also of $x^{29} + y^{29}$.

2. Find two factors of $x^{20} - y^{20}$; also of $x^{30} - y^{30}$.

3. Find all the factors of $x^4 - y^4$; also of $a^5 - b^5$.

4. Find all the factors of $x^{16} - y^{16}$; also of $x^{10} - y^{10}$.

5. Find all the factors of $x^{12} - y^{12}$.

6. Make a rule for reading at sight the following quotients, $(x^5 + y^5) \div (x + y)$ and $(x^5 - y^5) \div (x - y)$.

7. Does a similar rule apply to $(x^7 + y^7) \div (x + y)$ and $(x^7 - y^7) \div (x - y)$?

8. Factor $x^6 + y^6$. Show that $x^2 + y^2$ is one factor by substituting $-y^2$ for x^2 in $(x^2)^3 + (y^2)^3$.

9. Is $x^3 - y^3$ a factor of $x^{15} - y^{15}$? Why?

REVIEW AND SUMMARY

1. What is meant by *factoring?* Is $x(a + b) + y(a + b)$ factored? Why?

2. By what principle is the monomial factor removed from

$$ax + ay + az?$$

3. What are the characteristics of the trinomial squares:

$$a^{2n} + 2\,a^n b^n + b^{2n};\quad a^{2n} - 2\,a^n b^n + b^{2n}?$$

Are the following trinomials squares? If not, state why they are not. $x^2 + xy + y^2$; $x^4 + x^2y^2 + y^4$; $a^2 - 2\,ab - b^2$; $4\,a^2 + 4\,ab + 4\,b^2$.

4. What are the factors of the difference of two squares:

$$x^{2n} - y^{2n}?$$

Factor $x^6 - y^6$ as the difference of two squares

How can $$x^{4n} + x^{2n}y^{2n} + y^{4n}$$

be changed into the difference of two squares?

5. What are the factors of the sum of two cubes:

$$x^3 + y^3?$$

Can $x^6 + y^6$ be factored in this way? $x^9 + y^9$?

6. What are the factors of the difference of two cubes:

$$x^3 - y^3?$$

Can $x^6 - y^6$ be factored in this way? $x^9 - y^9$?

7. State the conditions under which

$$x^2 + px + q$$

can be factored by inspection. Can $x^2 - 10x + 16$ be so factored? $x^2 - 10x - 16$?

8. Tell how to decide whether

$$ax^2 + bx + c$$

can be factored by inspection. Can $6x^2 + 13xy + 6y^2$ be so factored?

9. What are the factors of

$$ax + ay + bx + by?$$

Can $x^2 + ax + bx + ab$ be factored in this way? Show that $x^3 - x - 3x^2 + 3$ can be factored in this way.

10. What are the factors of

$$x^2 + y^2 + z^2 + 2xy + 2xz + 2yz?$$

State the characteristics of the square of a trinomial.

11.* State the **remainder theorem.** For what values of n is $x^n + y^n$ divisible by $x + y$? by $x - y$? For what values of n is $x^n - y^n$ divisible by $x - y$? by $x + y$?

Why is this theorem also called the **factor theorem**?

12. If k is an integer, what kind of a number is $2k$? What kind of a number is $2k + 1$?

13.* Is $x - y$ a factor of $x^{2k} - y^{2k}$? of $x^{2k+1} - y^{2k+1}$?

14.* Is $x + y$ a factor of $x^{2k} + y^{2k}$? of $x^{2k+1} + y^{2k+1}$?

MISCELLANEOUS ORAL EXERCISES

Factor the following:

1. $a^2b^2 + a^3b + a^4$.
2. $a^2b^2 - c^2$.
3. $x^2 + 4x + 4$.
4. $a^2 - 6a + 9$.
5. $x^2 + 9x + 20$.
6. $a^2 - 5a + 6$.
7. $x^2 + 5x + 6$.
8. $x^2 + 11x + 30$.
9. $4x^2 - y^2$.
10. $x^2 + 7x + 6$.
11. $a^4 - 25$.
12. $2ab + b^2 + a^2$
13. $a^2 + 15a - 16$.
14. $x^2 - 11x + 30$.
15. $-2xy + y^2 + x^2$.
16. $4a^2 + b^2 + 4ab$.
17. $(a + b)^2 - c^2$.
18. $5x^2 + 3x^3 + x^4$.
19. $5x^5 - 3x^6 + 2x^8$.
20. $a^2 - 13a + 30$.
21. $a^2 + 13a + 30$.
22. $x^2 - (y + 3)^2$.
23. $(x - 1)^2 - y^2$.
24. $1 + 6a + 9a^2$.
25. $a^2 + 4ab + 4b^2$.
26. $x^3 + y^3$.
27. $x^3 - y^3$.
28. $1 - y^3$.
29. $y^3 - 1$.
30. $x^3 + 1$.

MISCELLANEOUS WRITTEN EXERCISES

Classify the following expressions according to the types for factoring, and find the factors:

1. $x^2 - 13x + 42$.
2. $1 - 8x^3$.
3. $x^2 + 17x + 72$.
4. $4x^2 + 9y^2 + 12xy$.
5. $4x^2 + 9y^2 - 12xy$.
6. $5x^2 + 4ax + 7xy$.
7. $2n^2 - 6nc - 3ny + 9cy$.
8. $(x + y)^2 - (y - 2x)^2$.
9. $a^3 + b^3$.
10. $9x^2 + y^4 + 6xy^2$.
11. $2y^2a^3 + 4ya^2 - 8ya$.
12. $x^2 - 15x + 36$.
13. $9x^2 + 36y^4 + 36xy^2$.
14. $9y - 9z - 2xy + 2xz$.
15. $a^3 - 1$.
16. $a^4 + b^4 + 2a^2b^2$.
17. $a^2 + 2ab + b^2 - c^4$.
18. $27a^3 - 125$.

19. $4a^2 + 4ab + b^2$.
20. $4a^2 + 9x^4 - 12ax^2$.
21. $1 + x^3$.
22. $2x^2 + 5x + 3$.
23. $36 + 4x^6 + 24x^3$.
24. $(x-1)^2 - (x+1)^2$.
25. $8 + 64a^6$.
26. $ac - ax - 4bc + 4bx$.
27. $27 - 216a^3$.
28. $3^3 + 6^3a^3$.
29. $25(x+1)^2 - 4$.
30. $5cx - 10c + 4dx - 8d$.
31. $4(x+2)^2 + y^2 + 4(x+2)y$.
32. $ra + 2rh - 5sa - 10sh$.
33. $-2a^2b + a^4 + b^2$.
34. $2ha - hb + 6a - 3b$.
35. $3(a+1)^3 + 4(a+1)^2 + a + 1$.
36. $(x+a)^2 - (x-a)^2$.
37. $15m^2 + 34m + 15$.
38. $3x^2 + 27x + 42$.
39. $x^4 + 49a^2 + 14ax^2$.
40. $27a^6 - a^3x^3$.
41. $3^3a^6 + a^3x^3$.
42. $8bd - 40be + 3cd - 15ce$.
43. $x^2 - x - 240$.
44. $(x+2)^2 - 4(x-2)^2$.
45. $x^4 + 9y^2 - 6x^2y$.
46. $4a^2 - 7ca^2 - 4d^2 + 7cd^2$.
47. $x^2 + 31x + 240$.
48. $18 - 27c + 16b - 24cb$.
49. $4 - (a^2 + b^2 - 2ab)$.
50. $10r + 3bs - 6br - 5s$.
51. $25 + 64x^6 + 80x^3$.
52. $1000 - x^3$.
53. $10^3 + x^3$.
54. $8a^3 + a^3b^3 + b^2a^2$.
55. $100 - 49x^4$.
56. $100 + 625 + 500$.
57. $a^2 - 17a + 72$.
58. $a^2 + 17a + 72$.
59. $a^2 + 16b^2 - 8ab$.
60. $x^6 - y^9$.
61. $4a^2 + 23a - 72$.
62. $x^4 + 15x^2 - 100$.
63. $9^3 + 8^3$.
64. $9x^4 + 16y^2 + 24x^2y$.
65. $1 - 1000 = 1 - 10^3$.
66. $16a^2b^2 + 24ab + 36b^3$.
67. $64 + 8 = 4^3 + 2^3$.
68. $16a^2b^2 + 9a^2c^2 + 24a^2bc$.
69. $a^2 + 4b^2 + 4ab - 4x^2$.
70. $a^3b^6 + c^3$.
71. $5x^3 + 10x^3y^2 + 30x^3y^4$.
72. $16a^2c^2 + 4c^2x^2 + 16ac^2x$.
73. $a^6y^3 - z^3$.
74. $x^4 - 7x^2 - 120$.
75. $9a^4b^2 - 12a^3b + 4a^2$.
76. $8ab + 27ab^7$.
77. $x^4 + 4x^3 + 4 - x^6$.
78. $1 - 125a^3b^6$.

79. $16 + 16\,ab + 4\,a^2b^2$.
80. $64\,a^3 + 8\,a^2b^3$.
81. $65\,r^2 + 8\,r - 1$.
82. $a^2 - 13\,a - 140$.
83. $x^6 + 17\,x^3 + 30$.
84. $25 - (a^4 - 2\,a^2b^3 + b^6)$.
85. $36\,a^2 - 29\,ab + 5\,b^2$.
86. $a^2 - a - 380$.
87. $24\,a^4c^4 + a^6 + 144\,c^8a^2$.
88. $9\,x^2 + 4\,y^4 - 12\,xy^2 - 16$.
89. $81 + 100\,x^8 - 180\,x^4$.
90. $a^4 + 27\,a^2 + 180$.
91. $a^4 + 3\,a^2 - 180$.
92. $a^4 - 3\,a^2 - 180$.
93. $144 - (a^4 + b^2 - 2\,a^2b)$.
94. $81\,a^2b^4 + 49\,c^2 - 126\,ab^2c$.
95. $12\,s^2 - 23\,st + 10\,t^2$.
96. $36\,x^4 + 12\,x^2y^4 + y^8$.
97. $16\,x^6 + 9\,y^4 + 24\,x^3y^2 - 49$
98. $y^2 + 35\,y + 300$.
99. $5\,y^2 - 80\,y + 300$.
100. $39\,x^4 - 16\,x^2 + 1$.
101. $ac - bc + ad - bd$.
102. $625 - (31 - 4\,a^2)^2$.
103. $z^3 + ya - y^3z^3 - ay^4$
104. $x^4 + 2\,x^2 + 1 - x^2$.
105. $60\,x^2 + 7\,xy - y^2$.
106. $x^2 - 20\,xy + 75\,y^2$.
107. $x^2 - 17\,x - 60$.
108. $36\,a^2b^4 + c^2b^4 + 12\,ab^4c$.
109. $4\,a^2 + 9\,b^4 + 12\,ab^2 - 16\,a^4$.
110. $100 - (16\,x^2 + y^6 - 8\,xy^3)$.
111. $6\,rd - 15\,re + 22\,cd - 55\,ce$.
112. $-112\,a^2c^3 + 49\,a^4 + 64\,c^6$.
113. $x^3 - 2\,x^2 + x - 2$.
114. $4\,x^2 + 9\,y^2 + z^2 - 12\,xy - 4\,xz + 6\,yz$.
115. $a^2b^2 + a^2c^2 + b^2c^2 - 2\,a^2bc + 2\,acb^2 - 2\,abc^2$.
116. $x^3 + 2\,x^2 - x - 2$.
117. $x^3 + 2\,x^2 - 5\,x + 2$.
118. $4\,a^2 - 12\,ab + 4\,ac + 9\,b^2 + c^2 - 6\,bc$.
119. $a^6 + 10\,a^3 + 9\,a^2 + 25 - 6\,a^4 - 30\,a$.
120. $x^3 - 3\,x^2 - x + 3$.
121. $x^3 + 3\,x^2 - x - 3$.

CHAPTER VIII

EQUATIONS SOLVED BY FACTORING

116. **Quadratic Equations.** The equations solved in the preceding chapters have all been of the *first degree*, that is, they have involved the first power of the unknown and no higher power, after all possible reductions have been made.

An equation which involves the second power, but no higher power of the unknown, is called a **quadratic equation.**

E.g. $x^2 + x = 30$ is a quadratic equation.

But $x^2 + 5x + 2 = x(x + 1)$ is of the first degree, since it may be reduced to $5x + 2 = x$.

117. One method of solving quadratic equations is now to be considered, namely, the **method by factoring.**

Example. Solve the equation $x^2 + (x+1)^2 = 61$.

Solution. By F, $x^2 + x^2 + 2x + 1 = 61.$ (1)

By F, S, $2x^2 + 2x = 60.$ (2)

By $D \mid 2$, $S \mid 30$ $x^2 + x - 30 = 0.$ (3)

Factoring the left member,

$$(x + 6)(x - 5) = 0. \quad (5)$$

This equation is satisfied by $x = 5$ since $(5 + 6)(5 - 5) = 11 \cdot 0 = 0$, and also by $x = -6$ since $(-6 + 6)(-6 - 5) = 0 \cdot (-11) = 0$

It thus appears that equation (4) has two solutions, namely, 5 and -6.

Each of these values also satisfies equation (1). Thus, $5^2 + (5+1)^2 = 61$, and $(-6)^2 + (-6+1)^2 = 61$.

In the above solution, equation (5) has one member zero and the other member is in the *factored form.* The solution, then, consists in finding the values of x which make *either factor equal to zero*, since we know that if one of two factors is zero, then the product is zero.

ORAL EXERCISES

1. What is the product of 3 and zero? of 6 and zero? of 10 and zero? of 275 and zero?

2. If one of two factors is zero, what is the product? Does it matter what the other factor is?

3. What is the value of $k(x-1)$ if $x=1$?

4. What is the value of $(x-1)(x-5)$ if $x=1$? If $x=5$?

5. If $x=2$, what is the value of $(x-2)(x^2+4x-8)$?

6. If $x=-3$, what is the value of $(x+3)(x^3-2x+7)$?

7. Find a value of x which makes $(x-3)(x+2)$ equal to zero. Does this value of x make both factors equal to zero?

8. Find a value of x which satisfies the equation

$$(x-7)(x^2+2x-3)=0;$$

also one which satisfies $(x+8)(x^3+x+4)=0$.

Suggestion. Find a value of x which makes the first factor zero in each case.

9. Find two values of x which satisfy $(x-3)(x+4)=0$, also two which satisfy $(x+8)(x-3)=0$.

10. Find two values of x which satisfy $5x(x+7)=0$. Does $x=0$ satisfy this equation?

11. Find two values of x which satisfy $(3x-2)(2x+5)=0$.

118. **Rule for Solving Equations by Factoring.** The method of solution suggested by the foregoing examples consists of three steps:

(1) *Transform the equation so that all terms are collected in the left member, with similar terms united, leaving the right member zero.*

(2) *Factor the expression on the left.*

(3) *Find the values of the unknown, by setting each factor in turn equal to zero and solving.*

ORAL EXERCISES

Find two solutions of each of the following equations.

1. $(x-1)(x-2)=0$.
2. $(x-4)(x-5)=0$.
3. $(x-3)(x-7)=0$.
4. $(x+3)(x+2)=0$.
5. $(x+1)(x+2)=0$.
6. $(x+1)(x-3)=0$.
7. $(x+2)(x-5)=0$.
8. $(x-3)(x+3)=0$.
9. $(x+4)(x+5)=0$.
10. $(x+5)(x-5)=0$.
11. $(x-\frac{1}{2})(x+1)=0$.
12. $(2x-1)(x+7)=0$.
13. $(3x-1)(2x+1)=0$.
14. $(2x-1)(x+12)=0$.
15. $(x-\frac{1}{3})(x+\frac{1}{2})=0$.
16. $(4x-1)(2x+1)=0$.
17. $(5x-1)(3x-1)=0$.
18. $(2x-3)(3x-2)=0$.

WRITTEN EXERCISES

Find two solutions for each of the following equations:

1. $x^2-3x+2=0$.
2. $x^2+7x=30$.
3. $a^2-11a=-30$.
4. $a^2+13a=30$.
5. $3x+x^2=20x-72$.
6. $17x+30=-x^2-40$.
7. $7x^2+2x=30x-21$.
8. $11x+3x^2=20$.
9. $a^2+10a+8=-3a-34$.
10. $a^2+3a=10a+18$.
11. $a^2+10a=-24-4a$.
12. $2x^2-6x=-40+12x$.
13. $x^2-16=0$.
14. $x^2-1=0$.
15. $x^2-x=0$.
16. $x^2+x=0$.
17. $16-5x+x^2=-2x^2-20x-2$.
18. $4x^2=25$.
19. $x^2+5x+4=0$.
20. $x^2-10x+16=0$.
21. $x^2+12x+6=5x-4$.
22. $2x^2-7x=60+7x$.
23. $60x+4x^2+144=8x$.
24. $18x=63-x^2$.
25. $24x^2=12x+12$.
26. $2x=63-x^2$.
27. $22x+x^2=363$.
28. $3x^2+7x=6$.
29. $2x^2=2-3x$.
30. $x-2=-3x^2$.
31. $x^2-10=3x$.

119. Equations of Higher Degree than the Second. It is sometimes possible by the preceding process to solve equations in which the exponent of the highest power of the unknown is greater than 2.

Example 1. Solve the equation $x^3 + 30x = 11x^2$. (1)

By S, $x^3 - 11x^2 + 30x = 0$. (2)

By § 95, $x(x^2 - 11x + 30) = 0$. (3)

By § 105, $x(x-5)(x-6) = 0$. (4)

(4) is satisfied if $x = 0$, and also if $x - 5 = 0$, and if $x - 6 = 0$.

Hence the roots are $x = 0$, $x = 5$, $x = 6$.

Example 2. Solve the equation $x(x+1)(x-2)(x+3) = 0$.

Any value of x which makes one of these factors zero reduces the product to zero and hence satisfies the equation. Hence the roots of the equation are

$$x = 0,\ x = -1,\ x = 2,\ x = -3.$$

Example 3. Solve the equation $x^3 - 6x^2 + 11x - 6 = 0$.

Solution. Using the remainder theorem we get,

$$x^3 - 6x^2 + 11x - 6 = (x-1)(x^2 - 5x + 6)$$
$$= (x-1)(x-2)(x-3) = 0.$$

Hence the roots are $x = 1$, $x = 2$, and $x = 3$.

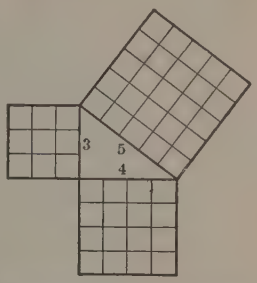
Notice that this process is applicable only when one member of the equation is *zero* and the other member can be factored.

WRITTEN EXERCISES

Solve the following equations by factoring:

1. $x^3 - x^2 = 6x$.
2. $5x = 4x^2 + x^3$.
3. $x^3 - 25x = 0$.
4. $x^3 - 3x^2 = -2x$.
5. $3x^3 = 15x^2 + 42x$.
6. $5x^3 + 315x = 80x^2$.
7. $x^2 + ax + bx + ab = 0$.
8. $x^2 - ax - bx + ab = 0$.
9. $4(x-2)^2 - (x+3)^2 = 0$.
10. $x^2 - ax + bx - ab = 0$.
11. $x^2 + ax - bx - ab = 0$.
12. $9(x+2)^2 - 4(x-3)^2 = 0$
13. $x^3 - x - 3x^2 + 3 = 0$.
14. $x^3 - 4x - 8x^2 + 32 = 0$.
15. $x^3 - 2x^2 - x + 2 = 0$.
16. $x^3 + 2x^2 - x - 2 = 0$.
17. $x^3 - 3x^2 - x + 3 = 0$.
18. $x^3 + 3x^2 - x - 3 = 0$.

120. Pythagorean Proposition. It is proved in geometry that if squares are constructed on the two sides, and also on the hypotenuse of a right-angled triangle, then the *sum of the squares on the sides is equal to the square on the hypotenuse.*



HISTORICAL NOTE

The Pythagorean Theorem. The theorem that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides was discovered by Pythagoras and is usually called the Theorem of Pythagoras. See next page. The Egyptians had discovered that, if the sides of a triangle are 3, 4, and 5, one angle is a right angle ($3^2 + 4^2 = 5^2$), and had used this in constructing right angles. Pythagoras extended the theorem to all right-angled triangles. The Theorem of Pythagoras is one of the most important in all mathematics.

The following example illustrates the use of this theorem:

Example. The difference between the two sides of a right triangle is 2 feet, and the length of the hypotenuse is 10 feet. Find the two sides.

Solution. Let x = the length of one side in feet.
Then $x + 2$ = the length of the other side.
Hence $x^2 + (x + 2)^2 = 10^2 = 100$. (Pythagorean theorem.)
Simplifying, $x^2 + 2x - 48 = 0$.
Factoring, $(x - 6)(x + 8) = 0$.
Hence $x = 6$ and $x = -8$.

It is clear that the length of a side of the triangle cannot be -8 feet. Hence 6 is the only one of these two results which has a meaning in this problem.

Therefore 6 and $6 + 2 = 8$ are the two sides of the triangle.

It happens frequently when a quadratic equation is used to solve a problem that one of the two numbers which satisfy the equation will not satisfy the conditions of the problem, as in this example.

PROBLEMS SOLVED BY FACTORING

In each of the following problems find all the solutions possible for the equations and then determine whether or not each solution has a reasonable interpretation in the problem.

1. The sum of the sides about the right angle of a right triangle is 35 inches, and the hypotenuse is 25 inches. Find the sides of the triangle.

2. The sum of the length and width of a rectangle is 17 rods, and the diagonal is 13 rods. Find the dimensions of the rectangle.

3. A room is 3 feet longer than it is wide, and the length of the diagonal is 15 feet. Find the dimensions of the room.

4. The length of the molding around a rectangular room is 46 feet, and the diagonal of the room is 17 feet. Find its dimensions.

5. The longest rod that can be placed flat on the bottom of a certain trunk is 45 inches. The trunk is 9 inches longer than it is wide. What are its dimensions?

6. The floor space of a rectanglar room is 180 square feet, and the length of the molding around the room is 56 feet. What are the dimensions of the room?

7. A rectangular field is 20 rods longer than it is wide, and its area is 2400 square rods. What are its dimensions?

8. A ceiling requires 24 square yards of paper, and the border is 20 yards long. What are the dimensions of the ceiling?

9. The area of a triangle is 18 square inches, and the sum of the base and altitude is 12 inches. Find the base and altitude.

10. The altitude of a triangle is 7 inches less than the base, and the area is 130 square inches. Find the base and altitude.

Pythagoras (569–500 B.C.), born on the Island of Samos, was the first of the great Greek mathematicians. He studied in Egypt, where no doubt he learned the practical geometry of the Egyptians. Later he returned to Samos to teach, but soon migrated westward to Sicily, and finally settled in the Greek colony of Croton in Southern Italy.

Here Pythagoras became the center of a widespread and influential organization, a sort of brotherhood for the moral education and purification of the community.

Pythagoras is famous for a system of Philosophy and for his studies in mathematics. The name *mathematics* and the name *philosophy* have been ascribed to him.

11. The sum of two numbers is 17, and the sum of their squares is 145. Find the numbers.

12. The difference of two numbers is 8, and the sum of their squares is 274. Find the numbers.

13. The difference of two numbers is 13, and the difference of their squares is 481. Find the numbers.

14. The sum of two numbers is 40, and the difference of their squares is 320. Find the numbers.

15. The sum of two numbers is 45, and their product is 450. Find the numbers.

16. The difference of two numbers is 32, and their product is 833. What are the numbers?

17. An open box is made from a piece of paper 20 inches square by cutting out a 5-inch square from each corner and turning up the sides. What is the volume of the box? What is the volume if the original square is x inches on a side?

18. An open box is made from a square piece of tin by cutting out a 5-inch square from each corner and turning up the sides. How large is the original square if the box contains 180 cubic inches?

If x = length of a side of the tin, then the volume of the box is: $5(x-10)(x-10) = 180$. (See the figure.)

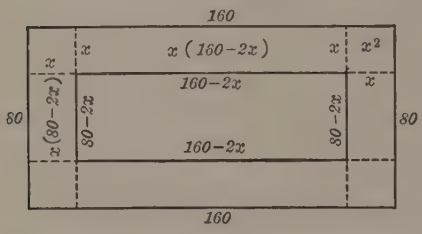
19. A rectangular piece of paper is 20 inches long and 16 inches wide. A box is made by cutting a 3-inch square out of each corner and turning up the sides. What is the volume of the box? What is the volume if the original paper is x inches wide and $x + 4$ inches long?

20. A rectangular piece of tin is 4 inches longer than it is wide. An open box containing 840 cubic inches is made by cutting a 6-inch square from each corner and turning up the ends and sides. What are the dimensions of the box?

21. The dimensions of a picture inside the frame are 8 by 10 inches. Find the area of the frame if its width is 2 inches. If its width is x inches.

22. The dimensions of a picture inside the frame are 12 by 16 inches. What is the width of the frame if its area is 288 square inches?

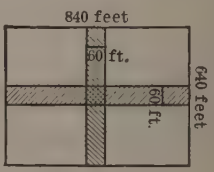
23. A farmer has a rectangular wheat field 160 rods long by 80 rods wide. In cutting the grain, he cuts a strip of equal width around the field. How many acres has he cut when the width of the strip is 8 rods?



24. How wide is the strip around the field of problem 23, if it contains $27\frac{1}{2}$ acres?

25. In the Northwest a farmer using a steam plow starts plowing around a rectangular field 640 by 320 rods. If the strip plowed the first day lacks 16 square rods of being 24 acres, how wide is it?

26. A rectangular piece of ground 840 by 640 feet is divided into 4 city blocks by two streets 60 feet wide running through it at right angles. How many square feet are contained in the streets?



27. A farmer lays out two roads through the middle of his farm, one running lengthwise of the farm and the other crosswise. How wide are the roads if the farm is 320 by 240 rods, and the area of the roads is 1671 square rods?

CHAPTER IX

COMMON FACTORS AND MULTIPLES

121. Common Factors. If an expression is a factor of each of two or more expressions, it is said to be a *common factor* of these expressions.

Thus, 8 is a common factor of 16 and 48, and 12 is a common factor of 12, 36, and 48.

If each of a given set of expressions is separated into prime factors, any common factor which they may have is at once apparent.

Illustrative Example. Find the common factors of

(1) $10(x+y)^2(x-y)$; (2) $5(x+y)(x^2-y^2)$; and (3) $15(x+y)(x^3-y^3)$.

Factoring, $10(x+y)^2(x-y) = 2 \cdot 5(x+y)(x+y)(x-y)$. (1)

$5(x+y)(x^2-y^2) = 5(x+y)(x+y)(x-y)$. (2)

$15(x+y)(x^3-y^3) = 3 \cdot 5(x+y)(x-y)(x^2+xy+y^2)$. (3)

The common *prime* factors are 5, $x+y$, and $x-y$. The highest common factor is the product of these common factors, namely, $5(x+y)(x-y)$.

122. Highest Common Factor. The product of all the common prime factors of two or more expressions is called their *highest common factor*. This is usually abbreviated to H. C. F.

The name *highest*, instead of *greatest*, common factor is used in algebra referring to the number of prime factors which enter. Thus, x^2 is of *higher degree* than x, although if $x = \frac{1}{2}$, x^2 is not greater than x.

ORAL EXERCISES

Find the H. C. F. of the following:

1. $3, 6, 12.$
2. $6, 48, 24.$
3. $a, a^2+a.$
4. $ab, ac, a^2.$
5. $3a^2, 2a^3, 6a.$
6. $x-y, x^2-y^2.$
7. $x+y, x^3+y^3.$
8. $x-y, x^3-y^3.$
9. $x^2-y^2, x^3-y^3.$
10. $x^2-y^2, x^3+y^3.$
11. $x^2-y^2, x^4-y^4.$
12. $a-b, a^2-2ab+b^2.$
13. $a^2-b^2, a^2+2ab+b^2.$
14. $a^3+b^3, a^2+2ab+b^2.$

WRITTEN EXERCISES

Find the H. C. F. of the following:

1. $x-y, x^2-y^2, x^2-2xy+y^2.$
2. $x^2+2x+1, 3x+6x^2+3x^3.$
3. $x^2+4x+4, x^2-6x-16.$
4. $x^2-8x+16, x^2+10x-56.$
5. $a^3-b^3, a^2-2ab+b^2.$
6. $x^3+y^3, x^2-y^2, x^2+2xy+y^2.$
7. $x^2-7x+12, ax-3a-bx+3b.$
8. $a^2-13a+42, a^3-216, a^2-a-30.$
9. $27+y^3, y^2+9y+18, y^2-9.$
10. $b^2+7b-30, b^2+11b-42, b^2-b-6.$
11. $a^3+2a^2+a, a^2+a, a^3+5a^2+4a.$
12. $x^3+y^3, x^3+x^2y+xy^2+y^3.$
13. $x^4+3x^3+2x^2, x^3+x^2, x^4+7x^3+6x^2.$
14. $x^2-11x+30, xz-5z+x^2-5x.$
15. $m^3-n^3, 2x^2m^2+2x^2mn+2x^2n^2.$
16. $x^2-1, x^3-1, x^2-13x+12.$
17. $1-64x^3, 1-16x^2, 5-2z-20x+8xz.$
18. $1+125a^3, 1+10a+25a^2, 1-25a^2.$

Find the H. C. F. of the following:

19. $ac - ax + 3bc - 3bx$, $a^3 + 27b^3$.
20. $5c - 2$, $5ac + 20c - 2a - 8$.
21. $4x^4 - x^2$, $2x^4 + x^3 - x^2$, $2x^4 - 3x^3 + x^2$.
22. $3a^3 - 3a$, $3a^3 - 6a^2 + 3a$, $6a^3 + 9a^2 - 15a$.
23. $6x - 10xy + 4xy^2$, $18x - 8xy^2$, $54x - 16xy^3$.
24. $16x^2 - 9y^2$, $12x^2 - 9xy$, $16xy - 12y^2$.
25. $36a^2 - 25b^2$, $18a^2 + 15ab$, $24ab + 20b^2$.
26. $3x^5 + 9x^4 - 3x^3$, $5x^2y^2 + 15xy^2 - 5y^2$, $7ax^2 + 21ax - 7a$.
27. $18x^3 - 57x^2 + 30x$, $9x^3 - 15x^2 + 6x$, $18x^3 - 39x^2 + 18x$.

COMMON MULTIPLES

123. **Multiples.** An algebraic expression is said to be a **multiple of any of its factors.** In particular, any expression is a multiple of itself and of one.

Thus, 18 is a multiple of 1, 2, 3, 6, 9, and 18, but not of 8 or of 12. $3a^2x^2$ is a multiple of 3, $3x$, $3x^2$, etc.

Since a multiple of an expression is divisible by that expression, it must contain as a factor every factor of that expression.

E.g. 108 is a multiple of 54 and contains as factors all the prime factors of 54; namely, 3, 3, 3, and 2.

124. **Common Multiples.** An expression is a **common multiple** of two or more expressions if it is a multiple of each of them. The **lowest common multiple** of a set of expressions is that one of their common multiples which contains the *smallest number* of prime factors. The lowest common multiple is usually abbreviated to L. C. M.

Thus, $x^2 - y^2$ is a common multiple of $x + y$ and $x - y$; also $x^4 - y^4$ is a common multiple of $x + y$ and $x - y$; but evidently $x^2 - y^2$ contains a smaller number of prime factors than $x^4 - y^4$, and hence is the *lowest common multiple.*

The process of finding the lowest common multiple of a set of expressions is shown in the following example:

Illustrative Example. Find the lowest common multiple of

(1) $x^2 - y^2$; (2) $x^2 + 2xy + y^2$; and (3) $x^2 - 2xy + y^2$.

Factoring, $x^2 - y^2 = (x - y)(x + y)$. (1)

$x^2 + 2xy + y^2 = (x + y)(x + y)$. (2)

$x^2 - 2xy + y^2 = (x - y)(x - y)$. (3)

In order that an expression may be a multiple of (1) it must contain the factors $x - y$ and $x + y$. To be a common multiple of (1) and (2) it must contain an additional factor $x + y$; that is, it must contain $(x - y)$, $(x + y)$, $(x + y)$.

To be a common multiple of (1), (2), and (3) it must contain an additional factor $x - y$; that is, it must contain the factors $(x - y)$, $(x + y)$, $(x + y)$, $(x - y)$.

The product $(x - y)(x + y)(x + y)(x - y) = (x - y)^2(x + y)^2$ is the lowest common multiple of (1), (2), and (3).

Rule for finding Lowest Common Multiple. *To obtain the lowest common multiple of a set of expressions:*

(1) *Find the prime factors of each expression.*

(2) *Use all factors of the first expression, together with those factors of the next expression which are not in the first, those of the third which are in neither the first nor in the second, etc.*

It is evident that in this manner we obtain a product which is a common multiple of the given expressions, but such that if any one of these factors is omitted, it will cease to be a multiple of some one of the expressions; that is, it will no longer be a common multiple of them *all*.

Thus, if in the example above either of the factors $x - y$ is omitted, the product will no longer be a multiple of $x^2 - 2xy + y^2$.

ORAL EXERCISES

Find the L. C. M. of the following sets of expressions:

1. 3, 5.	6. ab, bc, ac.	11. $2x + 3$, $x - 4$.
2. 3, 4, 6.	7. $3a$, $2b$, $4c$.	12. $m + 3$, $m - 5$.
3. 6, 48, 24.	8. $a + 1$, $a - 1$.	13. ax^2, b^2x, a^2b^2.
4. a, b.	9. $b + 2$, $b + 3$.	14. $x + 4$, $x - 1$.
5. a, bk.	10. $1 - a$, $1 - 2a$.	15. xyz, yzv, $2vx$

WRITTEN EXERCISES

Find the L. C. M. of the following expressions:

1. $2 \cdot 3 \cdot 4$; $3 \cdot 7 \cdot 8$; $2^3 \cdot 3 \cdot 4$.
2. $5\,x^2y^4$, $10\,x^3y$, $25\,x^2y$.
3. $2\,ab$, $6\,a^2$, $4\,b^2c$.
4. $x^2 - y^2$, $x^2 - 2\,xy + y^2$.
5. $x - y$, $x + y$, $x^2 - y^2$.
6. $4 - x^2$, $2 - x$, $2 + x$.
7. $a^2 + 2\,ab + b^2$, $a^2 - 2\,ab + b^2$.
8. $x^2 + 3\,x + 2$, $x^2 - 4$, $x^2 - 1$.
9. $25\,x^2 - 1$, $125\,x^3 - 1$.
10. $2\,x^2 - 7\,x + 6$, $4\,x^2 - 11\,x + 6$.
11. $x^3 - y^3$, $x - y$, $x^2 + xy + y^2$.
12. $x^3 - y^3$, $x^3 + y^3$, $x^2 - y^2$.
13. $5\,x^2 + 7\,x - 6$, $x^2 - 15\,x - 34$.
14. $x^3 + y^3$, $x^2 - y^2$, $(x - y)^2$.
15. $3\,abc$, $a^2 - 4\,ac + 4\,c^2$, $a - 2\,c$.
16. $x^2 - 1$, $x + 1$, $x^2 + 8\,x + 7$.
17. $4\,x^3y - 44\,x^2y + 120\,xy$, $3\,a^3x^2 - 22\,a^3x + 35\,a^3$.
18. $x^2 + 2\,xy + y^2$, $2\,ax^2 - 10\,ax + 12\,a$.
19. $3\,bx^2 - 21\,bx + 36\,b$, $x^2 - 5\,x + 4$.
20. $5\,a^2b^2 - 5\,a^2c^2$, $b^2 + 2\,bc + b + c + c^2$.
21. $15\,c^2ax^2 + 16\,c^2ax + c^2a$, $2\,cax^2 + 10\,cax + 8\,ca$.
22. $x^3 - 6\,x + 4$, $x - 4$, $x^2 - 16$.

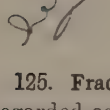
Solution. By means of the remainder theorem, we find,

$$x^3 - 6\,x + 4 = (x - 2)(x^2 + 2\,x - 2).$$

Hence the L. C. M. is

$$(x - 2)(x + 4)(x - 4)(x^2 + 2\,x - 2).$$

23. $x^3 - 6\,x^2 + 5$, $x + 3$, $x^2 - 1$.
24. $x^2 + 5\,x - 1$, $x^2 - 6\,x + 3$, $x - 1$.
25. $x^2 - 5\,x + 6$, $x^2 - 7\,x + 12$, $x^2 - 9\,x + 20$.



CHAPTER X

ALGEBRAIC FRACTIONS

125. Fractions. In arithmetic a fraction such as $\frac{2}{3}$ is usually regarded as 2 of the 3 equal parts of a unit.

However, a fraction such as $\frac{5}{3\frac{1}{2}}$ cannot be regarded in this way, since a unit cannot be divided into $3\frac{1}{2}$ equal parts. $\frac{5}{3\frac{1}{2}}$ indicates that 5 is to be divided by $3\frac{1}{2}$; *i.e.* $\frac{5}{3\frac{1}{2}} = 5 \div 3\frac{1}{2}$.

In algebra a fraction is usually regarded as an **indicated division** in which the numerator is the dividend and the denominator is the divisor.

Thus, $\frac{a}{b}$ is understood to mean $a \div b$.

Terms of a Fraction. The numerator and denominator are together called the *terms of the fraction*.

Operations on algebraic fractions are performed in accordance with the same rules as in arithmetic.

For example, just as $\frac{2}{3} = \frac{4}{6}$, so $\frac{a}{b} = \frac{2a}{2b}$, $\frac{m}{n} = \frac{mx}{nx}$,

and $$\frac{a+b}{c+d} = \frac{(x-y)(a+b)}{(x-y)(c+d)}.$$

ORAL EXERCISES

Supply the missing numerator or denominator in each of the following:

1. $\frac{1}{2} = \frac{\quad}{4}$.
2. $\frac{3}{4} = \frac{\quad}{12}$.
3. $\frac{1}{a} = \frac{\quad}{ka}$.
4. $\frac{a}{b} = \frac{\quad}{cb}$.
5. $\frac{m}{n} = \frac{\quad}{3n}$.
6. $\frac{1}{a} = \frac{\quad}{-2a}$.

7. $\frac{6}{9}=\frac{2}{\square}$.

8. $\frac{x^2}{xy}=\frac{\square}{y}$.

9. $\frac{ax+ay}{a^3}=\frac{\square}{a^2}$.

10. $\frac{15}{20}=\frac{\square}{4}$.

11. $\frac{kx}{xy}=\frac{\square}{y}$.

12. $\frac{x^3+y^3}{a(x+y)}=\frac{\square}{a^2}$.

13. $\frac{x^2-y^2}{a(x-y)}=\frac{\square}{a}$.

14. $\frac{1}{a-1}=\frac{\square}{1-a}$.

15. $\frac{1}{a+1}=\frac{\square}{(a+1)(a-1)}$.

16. $\frac{x^2+3x+2}{x^2-1}=\frac{x+2}{\square}$.

17. $\frac{x^3-y^3}{x^2-y^2}=\frac{\square}{x+y}$.

18. $\frac{1}{b+2}=\frac{\square}{(b+3)(b+2)}$.

19. $\frac{x^2-y^2}{(x+y)^2}=\frac{\square}{x+y}$.

20. $\frac{x^2-y^2}{(x-y)^2}=\frac{\square}{x-y}$.

The preceding examples illustrate

Principle XVII

126. **Rule.** *Both terms of a fraction may be multiplied or divided by the same number without changing its value.*

HISTORICAL NOTE

Fractional Notation. Our present method of representing fractions is the result of a long historical development. The **Babylonians** used 60ths only. Thus $\frac{1}{2}$ was called 30 sixtieths, $\frac{1}{3}$ was called 20 sixtieths, etc.

The **Egyptians** used only fractions with a numerator 1. The denominator only was written, a dot over it indicating that it represented a fraction. Thus $\dot{3}$ would mean $\frac{1}{3}$. The use of "unit" fractions only necessitated the reduction of other fractions to this form. Thus $\frac{2}{5}=\frac{1}{3}+\frac{1}{15}$, and $\frac{3}{7}=\frac{1}{4}+\frac{1}{28}$.

The **Greeks** used one accent to indicate the numerator and two to indicate the denominator. Thus $2'\ 3''$ meant $\frac{2}{3}$.

The **Romans** used a duodecimal system of fractions. That is, all their fractions were twelfths. The Hindus wrote $\frac{2}{3}$ in the form $\frac{2}{3}$, not using the bar. Alkarismi (see page 101) used the bar, writing the fraction in its present form. Decimal fractions came in much later than the common fractions. Simon Stevinus of Bruges in Belgium (1548–1620) was the first to treat decimal fractions systematically.

REDUCTION OF FRACTIONS TO LOWEST TERMS

127. By Principle XVII any factor common to the numerator and denominator of a fraction may be cancelled. That is

$$\frac{ak}{bk} = \frac{a}{b}.$$

Thus,
$$\frac{2 \cdot \cancel{3} \cdot 4 \cdot 5}{\cancel{3} \cdot 7 \cdot 11} = \frac{2 \cdot 4 \cdot 5}{7 \cdot 11}; \qquad \frac{\overset{2}{\cancel{2^3}} \cdot \overset{3}{\cancel{3^2}} \cdot \overset{4}{\cancel{4}}\, x}{\underset{2}{\cancel{2^4}} \cdot \underset{3}{\cancel{3}} \cdot \underset{4}{\cancel{4^2}}} = \frac{3x}{2 \cdot 4} = \frac{3x}{8};$$

$$\frac{x^2 - 7x + 12}{x^2 - 5x + 6} = \frac{\cancel{(x-3)}(x-4)}{(x-2)\cancel{(x-3)}} = \frac{x-4}{x-2}.$$

Lowest Terms. If the terms of a fraction have no common factor, the fraction is said to be reduced to its *lowest terms.*

ORAL EXERCISES

Reduce the following fractions to lowest terms:

1. $\frac{4}{6}$.
2. $\frac{14}{21}$.
3. $\frac{4}{2x+4}$.
4. $\frac{x}{x^3 - 3x^2 + x}$.
5. $\frac{x-1}{(x-3)(x-1)}$.
6. $\frac{x+2}{x^2+3x+2}$.
7. $\frac{x-2}{x^2-3x+2}$.
8. $\frac{x+1}{x^2+3x+2}$.
9. $\frac{x-y}{x^2-y^2}$.
10. $\frac{x-y}{x^2-2xy+y^2}$.
11. $\frac{x+y}{x^2+2xy+y^2}$.
12. $\frac{(x+y)^2}{(x+y)^3}$.
13. $\frac{x+3}{x^2+5x+6}$.
14. $\frac{x-5}{x^2-4x-5}$.
15. $\frac{x-4}{x^2-7x+12}$.
16. $\frac{x-1}{x^2-5x+4}$.
17. $\frac{x+1}{x^2+5x+4}$.
18. $\frac{x+1}{x^2+4x+3}$.
19. $\frac{x+3}{x^2+4x+3}$.
20. $\frac{x-2}{x^2-5x+6}$.
21. $\frac{x+2}{x^2+5x+6}$.
22. $\frac{x+2}{x^2+6x+8}$.
23. $\frac{x-2}{x^2-6x+8}$.
24. $\frac{x+3}{x^2+5x+6}$.
25. $\frac{x-3}{x^2-5x+6}$.
26. $\frac{x+3}{x^2+8x+15}$.
27. $\frac{x-3}{x^2-8x+15}$.

WRITTEN EXERCISES

Reduce the following fractions to lowest terms:

1. $\frac{3 \cdot 9^2 \cdot 2^6}{2^4 \cdot 5^3 \cdot 9^4}$.

2. $\frac{4\,a^4b^{12}c^3}{8\,a^3b^4c^4}$.

3. $\frac{x^2y^3z^4}{xy^2z^3}$.

4. $\frac{a^4b^3}{a^2b^2}$.

5. $\frac{x^2+2\,xy+y^2}{x^2-y^2}$.

6. $\frac{x^2+7\,x-30}{x^2-7\,x+12}$.

7. $\frac{x^3-y^3}{2\,x^2-3\,xy+y^2}$.

8. $\frac{64-b^3}{16-8\,b+b^2}$.

9. $\frac{x^3+27\,z^3}{xy-5\,x+3\,yz-15\,z}$.

10. $\frac{1-216\,c^3}{x-4\,y-6\,cx+24\,cy}$.

11. $\frac{14\,bz-2\,bx+ax-7\,az}{x^2-49\,z^2}$.

12. $\frac{3\,a^2-29\,a+56}{63-9\,a-7\,m+ma}$.

13. $\frac{a(x-y)^3}{(x^2-y^2)(x-y)}$.

14. $\frac{x^3+27}{4\,x^2+24\,x+36}$.

15. $\frac{a^2-3\,a-3\,b+ab}{(a^2-b^2)(a-3)}$.

16. $\frac{5\,x^3y^4-12\,x^2y^5+7\,x^3y^2}{6\,x^5y^2+3\,x^2y^2}$.

17. $\frac{5\,c+10\,b-bc-2\,b^2}{8\,c^3+64\,b^3}$.

18. $\frac{4\,x^4-28\,x^3+48\,x^2}{2\,x^4-8\,x^3+6\,x^2}$.

19. $\frac{9\,a^2b^4+18\,a^2b^3c+9\,a^2b^2c^2}{3\,ab^3-3\,abc^2}$.

20. $\frac{7\,xy^2-133\,xy+126\,x}{15\,xy^2-36\,xy+21\,x}$.

21. $\frac{20\,x^3+20\,x^2y+5\,xy^2}{60\,x^5-15\,x^3y^2}$.

22. $\frac{3\,ab^4-3\,ab^2c^2}{27\,a^3b^2+27\,a^3bc}$.

23. $\frac{4\,a^3-42\,a^2+20\,a}{2\,a^4b^6-20\,a^3b^6}$.

24. $\frac{(x-1)(x-2)(x-3)(x-4)}{(x-1)(x-3)(x-3)(x-4)}$.

25. $\frac{(x^2-y^2)(x^2+2\,xy+y^2)}{(x^2-2\,xy+y^2)(x+y)}$.

26. $\frac{(x^2-1)(x^2+1)(3\,x^2+3)}{3(x^4-1)}$.

27. $\frac{(x^2-4)(x^2+4)(x+5)}{(x-2)(x^2+3\,x-10)(x^4-16)}$.

28. $\frac{(2\,x^2+3\,x-2)(2\,x^2+7\,x+6)}{(4\,x^2+4\,x-3)(x^2+4\,x+4)}$.

29. $\frac{(2\,x^2-3\,x+1)(2\,x^2+3\,x+1)}{(x^2-1)(4\,x^2-1)}$.

30. $\frac{(2\,x^2+5\,x+2)(3\,x^2+10\,x+3)}{(3\,x^2+7\,x+2)(2\,x^2+7\,x+3)}$.

REDUCTION OF FRACTIONS TO A COMMON DENOMINATOR

128. Illustrative Examples. 1. Reduce $\frac{1}{2}$ and $\frac{1}{3}$ to a common denominator.

Solution. $\frac{1}{2}=\frac{3}{6}$; $\frac{1}{3}=\frac{2}{6}$. Is 6 a common multiple of 2 and 3? Is it the least common multiple of 2 and 3?

2. Reduce $\frac{1}{a}$ and $\frac{1}{b}$ to a common denominator.

Solution. $\frac{1}{a}=\frac{b}{ab}$; $\frac{1}{b}=\frac{a}{ab}$.

How is the common denominator related to the denominators a and b?

3. Reduce $\frac{2}{a+1}$, $\frac{3}{(a+1)(a+2)}$, $\frac{4}{a+2}$ to a common denominator.

Solution. The required denominator is the L. C. M. of the given denominators; that is, $(a+1)(a+2)$.

Hence, $\frac{2}{a+1}=\frac{2(a+2)}{(a+1)(a+2)}$ and $\frac{4}{a+2}=\frac{4(a+1)}{(a+1)(a+2)}$.

$\frac{3}{(a+1)(a+2)}$ already has the required denominator.

ORAL EXERCISES

Reduce each of the following to a common denominator:

1. $\frac{1}{a}, \frac{1}{b}$.
2. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.
3. $\frac{1}{a}, \frac{1}{-a}$.
4. $\frac{1}{ab}, \frac{1}{ac}$.
5. $\frac{a}{bc}, \frac{b}{ac}$.
6. $\frac{a}{xy}, \frac{b}{xz}$.
7. $\frac{1}{ab}, \frac{1}{bc}, \frac{1}{ac}$.
8. $\frac{a}{bc}, \frac{b}{ac}, \frac{c}{ab}$.
9. $\frac{m}{xy}, \frac{n}{xz}, \frac{s}{yz}$.
10. $\frac{1}{x+1}, \frac{1}{x+2}$.
11. $\frac{1}{a-3}, \frac{1}{a+4}$.
12. $\frac{a}{a-b}, \frac{c}{c-d}$.
13. $\frac{1}{1-k}, \frac{1}{k-1}$.
14. $\frac{1}{x+2}, \frac{1}{x-2}$.
15. $\frac{a}{a-b}, \frac{b}{a+b}$.

129. **By the formula** $\frac{a}{b} = \frac{ak}{bk}$, Principle XVII, both terms of a fraction may be multiplied by the same number.

In this manner any fraction may be changed into an equal fraction whose denominator is any given multiple of the denominator of the given fraction.

E.g. $\frac{3}{4} = \frac{3 \cdot 5}{4 \cdot 5}$, $\frac{a-b}{a+b} = \frac{(a-b)(a+b)}{(a+b)(a+b)} = \frac{a^2-b^2}{(a+b)^2}$.

Any two or more fractions may therefore be changed into equal fractions which shall have a *common denominator*, namely, a common multiple of the denominators of the given fractions.

Illustrative Example. Reduce

$$(1)\ \frac{x-1}{x+1};\ (2)\ \frac{x+1}{x-1};\ \text{and}\ (3)\ \frac{2x+3}{x^2-1}$$

to fractions having a common denominator.

The L. C. M. of the denominators is $(x-1)(x+1)$. Multiply the numerator and denominator of each fraction by an expression which will make the denominator of each new fraction $(x-1)(x+1)$.

Thus, $\frac{x-1}{x+1} = \frac{(x-1)(x-1)}{(x+1)(x-1)} = \frac{x^2-2x+1}{(x+1)(x-1)}$; (1)

$\frac{x+1}{x-1} = \frac{(x+1)(x+1)}{(x-1)(x+1)} = \frac{(x+1)^2}{(x+1)(x-1)}$; (2)

$\frac{2x+3}{x^2-1} = \frac{2x+3}{(x+1)(x-1)}$. (3)

It is best to *indicate* the multiplication in the common denominator, since this makes it more easily apparent by what expression the terms of a fraction must be multiplied in order to reduce it to a fraction with the required denominator.

ORAL EXERCISES

Supply the missing numerators:

1. $\frac{1}{x} = \frac{\qquad}{x^2(x-1)}$.

2. $\frac{x}{x+1} = \frac{\qquad}{x^2+8x+7}$.

3. $\frac{3}{x+5} = \frac{\qquad}{x^2+7x+10}$.

4. $\frac{2}{x-5} = \frac{\qquad}{x^2-2x-15}$.

130. The Three Signs of a Fraction. There are three signs in connection with a fraction: (1) the sign of the fraction itself, (2) the sign of the numerator, and (3) the sign of the denominator. Any two of these signs may be changed simultaneously without changing the value of the fraction.

Thus, (1) $\frac{a}{b}=\frac{-a}{-b}$. (2) $\frac{a}{b}=-\frac{a}{-b}$. (3) $\frac{a}{b}=-\frac{-a}{b}$.

If either the numerator or the denominator is a polynomial, its sign can be changed only by changing the sign of each term in it.

E.g. $\frac{a+b}{c-d}=\frac{-a-b}{-c+d}=-\frac{a+b}{-c+d}$.

If either the numerator or the denominator is in the factored form, its sign can be changed only by changing the sign of an *odd* number of its factors.

E.g. $\frac{(x+y)(x-y)}{(a+b)(a-b)}=\frac{(x+y)(y-x)}{(a+b)(b-a)}$.

This is useful in examples like the following:

Example. Reduce $\frac{x+1}{1-x}$, $\frac{x}{x^2-1}$, and $\frac{1}{x+1}$ to fractions having a common denominator.

$$\frac{x+1}{1-x}=\frac{-x-1}{x-1}=\frac{(x+1)(-x-1)}{(x+1)(x-1)}=\frac{-x^2-2x-1}{x^2-1}.$$

$$\frac{x}{x^2-1}=\frac{x}{x^2-1}, \text{ and } \frac{1}{x+1}=\frac{x-1}{(x+1)(x-1)}=\frac{x-1}{x^2-1}.$$

ORAL EXERCISES

Apply the sign changes shown in (1), (2), and (3) above to each of the following fractions:

(1) $\frac{a}{b-c}$. (4) $\frac{(a-b)(c-d)}{-x}$. (7) $-\frac{-c}{(d-a)(b-a)}$.

(2) $\frac{a-b}{c-d}$. (5) $\frac{1}{(1-x)(x+2)}$. (8) $\frac{-(a-b)(b-c)}{(c-d)(d-a)}$.

(3) $\frac{-a}{c+d}$. (6) $-\frac{b-a}{d-c}$. (9) $-\frac{(-a)(-b)(-c)}{d(-e)(-f)}$.

WRITTEN EXERCISES

Reduce each of the following sets of fractions to equivalent fractions having a common denominator:

1. $\frac{x+3}{x-y}, \frac{4}{x^2-2xy+y^2}$.

2. $\frac{a-1}{a^2-b^2}, \frac{a+1}{a^2+2ab+b^2}$.

3. $\frac{3-x}{x^2-9x+20}, \frac{x+4}{7x^2-26x-8}$.

4. $\frac{2}{3-x}, \frac{x-1}{x+1}, \frac{x+1}{x-3}$.

5. $\frac{a+b}{b-a}, \frac{a-b}{(a+b)^2}, \frac{a}{a^2-b^2}$.

6. $\frac{1}{x^2-3x-4}, \frac{1}{x^2+3x+2}$.

7. $\frac{a+1}{a^2-2ab+b^2}, \frac{a-1}{a^2+2ab+b^2}$.

8. $\frac{1}{a^3-b^3}, \frac{1}{b-a}, \frac{1}{a^2+ab+b^2}$.

9. $\frac{a}{5a^2-4a-12}, \frac{b}{a^2+4a-12}, \frac{c}{a-2}$.

10. $\frac{mr}{m-1}, \frac{d}{1+n}$.

11. $\frac{Rr}{(R+r)(m-1)}, \frac{1}{R+r}$.

12. $\frac{a}{n-a}, \frac{b}{n-b}$.

13. $\frac{W(T-Q)}{w(Q-t)}, \frac{V}{w}$.

14. $\frac{V}{V-v}, \frac{V}{V+v}, \frac{1}{V^2-v^2}$.

15. $\frac{RA}{a-A}, \frac{1}{R+A}, \frac{1}{A-a}$.

16. $\frac{R}{x}, \frac{r}{y}, \frac{1}{x-y}, \frac{Rx}{x+y}$.

131. **Reducing Integral Expressions to the Fractional Form.** Since any number may be written as a fraction with the denominator 1, the above process may be used to reduce an integral expression to the form of a fraction having any desired denominator.

Thus, $3=\frac{3\cdot 5}{5}$; $x-y=\frac{(x-y)(x^2-1)}{x^2-1}$, etc.

Illustrative Example. Reduce $5x$, $\frac{5x-1}{x^2-1}$, and $\frac{2x-y}{x-1}$ to fractions having a common denominator. The lowest common denominator is x^2-1.

Thus, $5x=\frac{5x(x^2-1)}{x^2-1}=\frac{5x^3-5x}{x^2-1}$, $\frac{5x-1}{x^2-1}=\frac{5x-1}{x^2-1}$, and

$$\frac{2x-y}{x-1}=\frac{(2x-y)(x+1)}{(x-1)(x+1)}=\frac{2x^2+2x-xy-y}{x^2-1}.$$

WRITTEN EXERCISES

Reduce the following expressions to fractions having a common denominator:

1. $5,\ x-y,\ \dfrac{5x-3}{x^2+2xy+y^2}$.

2. $\dfrac{3a-c}{x-y},\ \dfrac{2b-c}{x+y},\ 2c+2$.

3. $1+a+a^2,\ \dfrac{a+1}{a-1}$.

4. $x^2+xy+y^2,\ \dfrac{x+y}{x-y}$.

5. $x^2-xy+y^2,\ x^2-y^2,\ \dfrac{1}{x+y}$.

6. $x^4+x^2y^2+y^4,\ \dfrac{x}{x-y},\ \dfrac{y}{x+y}$.

7. $3a-2b-c,\ \dfrac{5}{a-b},\ \dfrac{2}{b-c}$.

8. $x^4-1,\ x^2-1,\ \dfrac{x+1}{x-1}$.

9. $x^2+2xy+y^2,\ \dfrac{1}{x+y},\ \dfrac{1}{x-y}$.

10. $x+y,\ x-y,\ \dfrac{x-y}{x^2+y^2},\ \dfrac{x+1}{x-y}$.

11. $\dfrac{RA}{a-A},\ r$.

12. $\dfrac{RA}{a-A},\ R-r$.

13. $T,\ \dfrac{Q+t}{2}$.

ADDITION AND SUBTRACTION OF FRACTIONS

132. Rule for Adding or Subtracting Fractions. By Principle III, § 16, we know that

$$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c} \text{ and } \frac{a-b}{c}=\frac{a}{c}-\frac{b}{c}.$$

Reading this in the reverse order we have the formula for adding and subtracting fractions:

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \text{ and } \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}.$$

From this formula we get the following rule.

(1) *Reduce the fractions to a common denominator.*

(2) *To add the fractions, add the numerators and place the sum over the common denominator.*

(3) *To subtract the fractions, subtract the numerators and place the remainder over the common denominator.*

(4) *Reduce the resulting fraction in each case to its lowest terms.*

Example. Add $\frac{1}{a+1}$ and $\frac{2}{a+3}$.

Solution. $\frac{1}{a+1}=\frac{a+3}{(a+1)(a+3)}$; $\frac{2}{a+3}=\frac{2(a+1)}{(a+1)(a+3)}$.

Hence, $\frac{1}{a+1}+\frac{2}{a+3}=\frac{a+3}{(a+1)(a+3)}+\frac{2(a+1)}{(a+1)(a+3)}$

$$=\frac{a+3+2(a+1)}{(a+1)(a+3)}=\frac{3a+5}{(a+1)(a+3)}.$$

ORAL EXERCISES

Perform the following additions:

1. $\frac{1}{a}+\frac{1}{b}$.
2. $\frac{1}{ab}+\frac{1}{ac}$.
3. $\frac{1}{a+1}+\frac{1}{a-1}$.
4. $\frac{1}{x+1}+\frac{1}{x+2}$.
5. $\frac{2}{a}+\frac{1}{-a}$.
6. $\frac{1}{1-k}+\frac{1}{k-1}$.
7. $\frac{1}{n-4}+\frac{1}{n-1}$.
8. $\frac{1}{x+2}+\frac{1}{x-2}$.

Example. From $\frac{3}{x-4}$ subtract $\frac{2}{x+3}$.

Solution.

$$\frac{3}{x-4}-\frac{2}{x+3}=\frac{3(x+3)}{(x-4)(x+3)}-\frac{2(x-4)}{(x-4)(x+3)}$$

$$=\frac{3(x+3)-2(x-4)}{(x-4)(x+3)}=\frac{x+17}{(x-4)(x+3)}.$$

ORAL EXERCISES

Perform the following subtractions:

1. $\frac{1}{a}-\frac{1}{b}$.
2. $\frac{1}{ab}-\frac{1}{ac}$.
3. $\frac{1}{a+1}-\frac{1}{a-1}$.
4. $\frac{1}{x+1}-\frac{1}{x+2}$.
5. $\frac{2}{-a}-\frac{1}{a}$.
6. $\frac{1}{1-k}-\frac{1}{k-1}$.

Example. Add $\dfrac{a-b}{a+b}$ and $\dfrac{a^2+2ab+b^2}{a^2-2ab+b^2}$.

Solution. The L. C. M. of the denominators is

$$(a-b)(a-b)(a+b).$$

$$\frac{a-b}{a+b}=\frac{(a-b)(a-b)(a-b)}{(a+b)(a-b)(a-b)}=\frac{a^3-3a^2b+3ab^2-b^3}{(a+b)(a-b)(a-b)},$$

and $$\frac{a^2+2ab+b^2}{a^2-2ab+b^2}=\frac{(a+b)(a^2+2ab+b^2)}{(a+b)(a-b)(a-b)}=\frac{a^3+3a^2b+3ab^2+b^3}{(a+b)(a-b)(a-b)}.$$

Adding the numerators, we have $2a^3+6ab^2$; whence the sum of the fractions is

$$\frac{2a^3+6ab^2}{(a+b)(a-b)(a-b)}.$$

WRITTEN EXERCISES

Perform the following additions and subtractions:

1. $\dfrac{1}{x-2}-\dfrac{1}{x+1}$.
2. $\dfrac{1}{a+4}-\dfrac{2}{a-3}$.
3. $\dfrac{1}{x-1}-\dfrac{1}{1-x}$.
4. $\dfrac{1}{x}-\dfrac{1}{y}$.
5. $\dfrac{a}{a-b}-\dfrac{b}{a+b}$.
6. $\dfrac{3}{k-1}-\dfrac{2}{k+1}$.
7. $\dfrac{4}{c-2}-\dfrac{3}{(c-1)(c-2)}$.
8. $\dfrac{1}{a}-\dfrac{1}{b}+\dfrac{1}{c}$.
9. $\dfrac{1}{ab}+\dfrac{1}{bc}-\dfrac{1}{abc}$.
10. $\dfrac{3}{a+b}-\dfrac{2}{a-b}+\dfrac{1}{a}$.
11. $\dfrac{a}{1-a}+\dfrac{1}{a-1}+\dfrac{1}{a}$.
12. $\dfrac{3}{x-y}-\dfrac{1}{y}+\dfrac{1}{x}$.
13. $\dfrac{x}{x-y}+\dfrac{x}{y}-\dfrac{1}{y}$.
14. $\dfrac{1}{x}+\dfrac{a}{y}-\dfrac{b}{z}$.
15. $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}$.
16. $\dfrac{a}{(x-1)(x+3)}-\dfrac{b}{x+3}$.
17. $\dfrac{1}{x+y}+\dfrac{1}{x-y}+\dfrac{1}{(x+y)(x-y)}$.
18. $\dfrac{1}{a+1}+\dfrac{1}{a-1}-\dfrac{1}{(a+1)(a-1)}$.
19. $\dfrac{3}{4}+\dfrac{2}{7}+\dfrac{a+b}{3a-b}$.
20. $\dfrac{x-y}{(x+y)^2}-\dfrac{x+y}{x^2-y^2}$.

21. $\dfrac{x^2-9x+18}{x^2-13x+36}+\dfrac{x}{4-x}$.

22. $\dfrac{2}{3}+\dfrac{a}{a+b}+\dfrac{b}{a-b}$.

23. $\dfrac{3}{2^3\cdot 3^2}+\dfrac{5}{2^2\cdot 3^4}-\dfrac{2}{2^4\cdot 3^3}$.

24. $\dfrac{a^2-9b^2}{a^2+6ab+9b^2}-\dfrac{a^2-6ab}{a^2-9b^2}$.

25. $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}-2$.

26. $\dfrac{2x^2}{x^2-y^2}-\dfrac{y}{x-y}-\dfrac{y}{x+y}$.

27. $\dfrac{a+1}{a^2+a+1}-\dfrac{a-1}{a^2-a+1}$.

28. $\dfrac{b}{1-b}+\dfrac{b}{1+b}-\dfrac{b^2}{1-b^2}$.

29. $\dfrac{x+1}{x-2}+\dfrac{x+1}{x+2}+\dfrac{3x+2}{x^2-4}$.

30. $\dfrac{x-1}{x+1}-\dfrac{x+1}{x-1}+\dfrac{x^2-5}{x^2-1}$.

31. $\dfrac{y^2}{y^2-1}+\dfrac{y}{y+1}-\dfrac{y}{1-y}$.

32. $\dfrac{1}{x}-\dfrac{1}{y}-\dfrac{1}{x-y}+\dfrac{-1}{x+y}$.

33. $\dfrac{1}{2}-\dfrac{b}{b-a}+\dfrac{2a^2}{b^2-a^2}+\dfrac{-a}{b+a}$.

34. $\dfrac{x^2+4xy}{x^3+y^3}+\dfrac{1}{x+y}-\dfrac{x}{x^2-xy+y^2}$.

35. $\dfrac{a-3}{a^2-3a+2}-\dfrac{a-1}{a^2-5a+6}+\dfrac{a}{a^2-4a+3}$.

36. $\dfrac{x}{x^2-5x-14}+\dfrac{2}{x-7}-\dfrac{x}{x^2-9x+14}$.

37. $\dfrac{a}{ac+ad-bc-bd}-\dfrac{b}{a^2-2ab+b^2}$.

38. $\dfrac{1}{y^2+8y+16}-\dfrac{1}{y(y+4)}+\dfrac{4}{y^2(y+4)}$.

39. $\dfrac{a+2}{a^2-a-6}+\dfrac{a-4}{a^2-7a+12}-\dfrac{a+2}{a^2-2a-8}$.

40. $\dfrac{2}{x^2-11x+30}-\dfrac{1}{x^2-36}+\dfrac{1}{x^2-25}$.

41. $\dfrac{1}{(x-1)(x+2)}+\dfrac{1}{(x+2)(x-3)}+\dfrac{1}{(x-1)(3-x)}$.

42. $\dfrac{1}{(a-b)(b-c)}-\dfrac{1}{(b-a)(c-d)}+\dfrac{1}{(b-c)(c-d)}$.

43. $\dfrac{4}{a-3}-\dfrac{a-1}{a^2+3a+9}+\dfrac{a^2-11a-3}{a^3-27}$.

MULTIPLICATION OF FRACTIONS

133. Fractions are multiplied as in arithmetic.

Thus $$\frac{2}{3}\times\frac{5}{7}=\frac{2\cdot 5}{3\cdot 7}=\frac{10}{21}.$$

Similarly,

$$\frac{a}{b}\times\frac{c}{d}=\frac{a\cdot c}{b\cdot d}=\frac{ac}{bd}.$$

That is, we have the following

Rule. *The product of two algebraic fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.*

134. Steps in Multiplying Fractions. The steps in multiplying one fractional expression by another are:

(1) *Reduce mixed expressions to fractions.*

(2) *Factor each numerator and each denominator.*

(3) *Cancel all factors common to numerators and denominators.*

(4) *Multiply the remaining factors in the numerators for the numerator of the product, and those of the denominators for the denominator of the product.*

These steps are illustrated in the following examples.

(1) $\frac{3}{4}\times\frac{8}{9}=\frac{\cancel{3}\cdot\overset{2}{\cancel{8}}}{\underset{3}{\cancel{4}}\cdot\cancel{9}}=\frac{2}{3}$; (2) $1\frac{2}{3}\times 2\frac{2}{5}=\frac{5}{3}\times\frac{12}{5}=\frac{\cancel{5}\cdot\overset{4}{\cancel{12}}}{\cancel{3}\cdot\cancel{5}}=4.$

(3) $\left(1+\frac{a}{b}\right)\times\frac{b^2}{b+a}=\frac{b+a}{b}\times\frac{b^2}{b+a}=\frac{\overset{b}{\cancel{b^2}}\cancel{(b+a)}}{\cancel{b}\cancel{(b+a)}}=b.$

(4) $$\frac{4x+12}{x^2+x-2}\times\frac{3x-3}{4x^2+4x-24}=\frac{\cancel{4}(x+3)\cdot 3(x-1)}{(x-1)(x+2)\cdot\cancel{4}(x+3)(x-2)}$$
$$=\frac{3}{(x+2)(x-2)}=\frac{3}{x^2-4}.$$

ORAL EXERCISES

Perform the following multiplications:

1. $\frac{ab}{c} \times \frac{c}{a}$.
2. $\frac{5\,ab}{2\,cx} \times \frac{4\,c^2x^2}{10\,a}$.
3. $\frac{abc}{xy} \times \frac{bx}{ac}$.
4. $\frac{cx^2}{by^2} \times \frac{cy}{dx}$.
5. $\frac{mxy}{nyc} \times \frac{ny}{mx}$.
6. $\frac{rst}{xy} \times \frac{xy^2}{r^2s}$.
7. $(a+b) \times \frac{1}{a^2-b^2} = \frac{a+b}{(a+b)(a-b)} = \frac{1}{a-b}$.
8. $(a-b) \times \frac{1}{a^2-b^2}$.
9. $\frac{a-b}{a+b} \times \frac{a^2-b^2}{a-b}$.
10. $(a-1) \times \frac{1}{a^3-1}$.
11. $(x+y) \times \frac{1}{x^3+y^3}$.
12. $\frac{x+y}{x-y} \times \frac{x-y}{x^3+y^3}$.
13. $\left(1+\frac{1}{x}\right) \times \frac{a}{x+1}$.
14. $\frac{2}{3(n+1)} \times 6(n+1)$.
15. $\frac{2}{(a-2)(a+3)} \times 3(a-2)$.

WRITTEN EXERCISES

Find the following indicated products and reduce the fractions to simplest forms:

1. $(1-a) \times \frac{1+a+a^2}{a-1}$.
2. $(x^3-y^3) \times \frac{x+y}{x-y}$.
3. $(x^2-2xa+a^2) \times \frac{x+a}{x-a}$.
4. $\frac{3x-1}{x^2-5x+6} \times (x^2-11x+18)$.
5. $\frac{a^2-4a+3}{a^2-8a+16} \times (a^2-5a+4)$.
6. $(x^2+9x+18) \times \frac{x-5}{x^2-2x-15}$.
7. $(1-x^3) \times \frac{1-x}{1+x+x^2}$.
8. $(27a^3-1) \times \frac{a+1}{9a^2+3a+1}$.

Perform the following multiplications:

9. $(a^2+ab+b^2)\times\dfrac{a-b}{a^3-b^3}$.

10. $(1-a+a^2)\times\dfrac{a+1}{a^3+1}$.

11. $\dfrac{x+1}{x^4-1}\times\dfrac{x^2-1}{(x+1)^2}$.

12. $\dfrac{a+b}{a^6-b^6}\times\dfrac{a^3+b^3}{a^2-b^2}$.

13. $\dfrac{3c}{(3-c)(3+c)}\times\dfrac{(c-3)(c+3)}{6cd}$.

14. $\dfrac{3a+b}{2+3x}\times\dfrac{(3x+2)(2a-b)}{(a+b)(3a+b)}$.

15. $\dfrac{a-b}{a^6-b^6}\times\dfrac{a^3-b^3}{a^2-b^2}$.

16. $\dfrac{(a+2b)(a+2b)}{a-2b}\times\dfrac{(2a-b)(a-2b)}{(3a+b)(a+2b)}$.

17. $\dfrac{3x^2y^2}{2yz^2}\times\dfrac{6az}{9x^3}$.

18. $\dfrac{5a(a-b)}{3c(a+b)}\times\dfrac{9(a+b)^2}{15(a^2-b^2)}$

19. $\dfrac{12c^3b}{5(c^3-b^3)}\times\dfrac{35(c^2+cb+b^2)}{14c^2b^2}$.

20. $\dfrac{y^2+3y+2}{y^2-5y+6}\times\dfrac{y^2-7y+12}{y^2+8y+7}$.

21. $\dfrac{x^2-x}{x^2-1}\times\dfrac{2x^2+4x+2}{3x^2+6x}$.

22. $\dfrac{a^2-10a+16}{a^2+6a+9}\times\dfrac{a+3}{a^2-4}$.

23. $\dfrac{(x+y)^2-z^2}{x^2+xy-xz}\times\dfrac{x}{(x-z)^2-y^2}\times\dfrac{(x-y)^2-z^2}{xy-y^2+yz}$.

24. $\dfrac{3(x+4)^2}{4(x+4)(x-7)}\times\dfrac{(x-7)^2}{3(x+4)(x-7)}$.

25. $\dfrac{a^3+5a^2-36a}{a^2-7a-144}\times\dfrac{(a-16)(a-3)}{a(a-4)(a+2)}$.

26. $\dfrac{3a(a+7)(a-5)}{7b(a+3)(a+7)}\times\dfrac{b(a+3)(a+10)}{a(a-5)(a-10)}$.

27. $\dfrac{3t^2-2t-1}{2t^2+t-1}\times\dfrac{2t^2+5t-3}{3t^2+7t+2}\times\dfrac{4t^2+10t+4}{4t^2-2t-2}$.

28. $\dfrac{6x^2-7x+2}{10x^2-7x+1}\times\dfrac{6x^2-5x-1}{6x^2+x-1}\times\dfrac{10x^2+3x-1}{5x^2-4x-1}$.

DIVISION OF FRACTIONS

135. Fractions are divided by the same rule as in arithmetic.

Thus, $$\frac{2}{3} \div \frac{8}{9} = \frac{2}{3} \times \frac{9}{8} = \frac{2 \cdot \overset{3}{\cancel{9}}}{\cancel{3} \cdot \underset{4}{\cancel{8}}} = \frac{3}{4}.$$

Similarly in algebra, $$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

That is, we have the following

Rule. *To divide by a fraction, invert the divisor and multiply by the fraction thus obtained.*

Steps in Dividing Fractions. The steps in dividing by a fraction are:

(1) *Reduce all mixed expressions to the fractional form.*
(2) *Factor each numerator and each denominator.*
(3) *Invert the fraction by which you are dividing.*
(4) *Now proceed as in multiplying fractions.*

These steps are illustrated in the following examples.

(1) $$\frac{ab}{x^2 - 3x + 2} \div \frac{b}{x - 1} = \frac{ab\cancel{(x-1)}}{\cancel{(x-1)}(x-2)} \times \frac{\cancel{x-1}}{\cancel{b}} = \frac{a}{x-2}.$$

(2) $$\frac{ax - ay}{b(x+y)} \div (x - y) = \frac{a\cancel{(x-y)}}{b(x+y)} \times \frac{1}{\cancel{x-y}} = \frac{a}{b(x+y)}.$$

(3) $$\left(1 + \frac{b}{a}\right) \div (a^2 - b^2) = \frac{\cancel{a+b}}{a} \times \frac{1}{\cancel{(a+b)}(a-b)} = \frac{1}{a(a-b)}.$$

In Example (3), the divisor is an integral expression, $a^2 - b^2$. This may be written $\frac{a^2 - b^2}{1}$, and this inverted is $\frac{1}{a^2 - b^2}$.

ORAL EXERCISES

Perform the following divisions:

1. $\frac{a}{b} \div \frac{a^2}{b^2}$.
2. $\frac{x^2}{y^2} \div \frac{x}{y}$.
3. $\frac{x^3}{y^3} \div \frac{x^2}{y^2}$.
4. $\frac{x^3}{y^5} \div \frac{x^4}{y^3}$.
5. $\frac{a^4}{b^4} \div \frac{a^3}{b^5}$.
6. $\frac{a^5}{b^3} \div \frac{a^4}{b^4}$.

WRITTEN EXERCISES

Perform the following divisions:

1. $\frac{ac}{b} \div c.$
2. $\frac{ab}{c} \div \frac{bx}{c}.$
3. $\frac{abc}{xy} \div \frac{bc}{ax}.$
4. $\frac{ax^2}{by^2} \div \frac{ax}{cy}.$
5. $\frac{nxy}{my} \div \frac{ny}{mx}.$
6. $\frac{rst}{xy} \div \frac{r^2s}{x^2y}.$
7. $\frac{abc}{x-1} \div ab.$
8. $\frac{xy}{a-b} \div xy.$
9. $\frac{a+1}{a-2} \div \frac{3(a+1)}{a+4}.$
10. $\frac{24\,ab}{25\,c} \div \frac{16\,ab}{15}.$
11. $\frac{4\,k^2}{-9\,abc^2} \div -\frac{2\,k}{ab}.$
12. $\frac{ab^2}{(x+y)^2} \div \frac{ab}{x+y}.$

Find the following indicated quotients and reduce the fractions to their lowest terms:

13. $\frac{x^3-y^3}{x+y} \div (x^2+xy+y^2).$
14. $\frac{x^3+y^3}{x-y} \div (x^2-y^2).$
15. $\frac{x^2+4x+4}{x^2-2x+1} \div (x^2-4).$
16. $\frac{1-27x^3}{1-9x^2} \div (1+3x+9x^2).$
17. $\frac{x^2+2x-35}{x^2+10x+21} \div (x^2-4x-5).$
18. $\frac{x^2-16x+39}{x^2-8x+15} \div (x^2-x-156).$
19. $\frac{ac+bc-ad-bd}{xy-4x-3y+12} \div (cx-3c-dx+3d).$
20. $\frac{x^2+ax+bx+ab}{x^2+bx-3x-3b} \div (x^2+ax-5x-5a).$
21. $\frac{mr+ms-nr-ns}{mx-m-nx+n} \div (3r-xr+3s-xs).$
22. $\frac{x^2-3x-88}{x^2-9x-22} \div (x^2+9x+8).$

EXERCISES IN MULTIPLICATION AND DIVISION

Perform the following indicated operations:

1. $\dfrac{a^3+b^3}{a^2-9b^2} \times \dfrac{a+3b}{a+b}$.

2. $\dfrac{x^2+x-2}{x^2-3x} \div \dfrac{x^3+2x^2}{x^2+9x-36}$.

3. $\dfrac{x^2-6x-16}{x^2+4x-21} \div \dfrac{x^2+9x+14}{x^2-8x+15}$.

4. $\dfrac{x^2-1}{x^2-4x-5} \times \dfrac{x^2-25}{x^2+2x-3}$.

5. $\dfrac{x^2+9xy+18y^2}{x^2-9xy+20y^2} \div \dfrac{x^2+6xy+9y^2}{xy^2-4y^3}$.

6. $\dfrac{3a^4-9a^3-54a^2}{9a^3-117a^2+378a} \div \dfrac{a^3+8a^2+15a}{3a^2-33a+84}$.

7. $\dfrac{a^2-11a+30}{a^3-6a^2+9a} \times \dfrac{a^2-3a}{a^2-25} \div \dfrac{a^2-9}{a^2+2a-15}$.

8. $\dfrac{x^2-10x+21}{x^2+x-56} \div \dfrac{x^2-8x+15}{x^2+4x-32}$.

9. $\dfrac{a^2-b^2}{ab^2x} \times \dfrac{b(a-b)}{a^2+2ab+b^2} \div \dfrac{a^2-2ab+b^2}{b(a+b)}$.

10. $\dfrac{a+b}{ab} \times \dfrac{a^2-b^2}{3(a^2+b^2)} \div \dfrac{(a-b)^2(a+b)^2}{3a^3y+3ay^3}$.

11. $\dfrac{5x^4-5x^3}{7x^2-56x-63} \div \dfrac{x^4-9x^3+8x^2}{14x^2+14x-1260}$.

12. $\dfrac{8y^2(y+4)(y+5)}{2^4(y+5)(y-7)} \div \dfrac{y(y+4)(y+8)}{2^3(y-7)(y+11)}$.

13. $\dfrac{x^3(x-2)^2}{(x+2)^2} \times \dfrac{(x+2)(x-3)}{(x-2)(x-7)} \div \dfrac{x^2(x-3)(x-5)}{(x-5)(x-7)}$.

14. $\dfrac{a^2b^2(c+5)(c-4)}{(c-4)(c-8)} \times \dfrac{(c-8)(c+9)}{(c+4)(c+7)} \div \dfrac{ab(c+9)(c-1)}{(c+7)(c+1)}$.

15. $\dfrac{21x^2+23x-20}{10x^2-27x+5} \times \dfrac{6x^2-11x-10}{3x^2+2x-5} \div \dfrac{7x^2+17x-12}{5x^2+9x-2}$.

16. $\dfrac{4b^2-17b+4}{6b^2-7b+2} \times \dfrac{10b^2-21b+9}{5b^2-23b+12} \times \dfrac{3b^2-5b+2}{4b^2-5b+1}$.

COMPLEX FRACTIONS*

136. **A complex fraction** is one which contains one or more fractions in its numerator or denominator or in both.

E.g. $\dfrac{1+\dfrac{1}{a}}{1-\dfrac{1}{a}}$ and $\dfrac{\dfrac{1}{x+1}+\dfrac{1}{x-1}}{\dfrac{1}{x-1}-\dfrac{1}{x+1}}$.

Simplifying a Complex Fraction.

A complex fraction is said to be *simplified* when it is reduced to an equal fraction, in its lowest terms, whose numerator and denominator are in the integral form. The following examples show how such reductions are made:

1. $$\frac{1+\dfrac{1}{a}}{1-\dfrac{1}{a}}=\frac{\dfrac{a+1}{a}}{\dfrac{a-1}{a}}=\frac{a+1}{a}\div\frac{a-1}{a}=\frac{a+1}{\cancel{a}}\times\frac{\cancel{a}}{a-1}=\frac{a+1}{a-1}.$$

This result may also be obtained directly by multiplying both terms of the given fraction by a which is the L. C. D. of all the small fractions in the complex fraction. This gives

$$\frac{1+\dfrac{1}{a}}{1-\dfrac{1}{a}}=\frac{a\left(1+\dfrac{1}{a}\right)}{a\left(1-\dfrac{1}{a}\right)}=\frac{a+\dfrac{a}{a}}{a-\dfrac{a}{a}}=\frac{a+1}{a-1}.$$

2. $$\frac{\dfrac{1}{x+1}+\dfrac{1}{x-1}}{\dfrac{1}{x-1}-\dfrac{1}{x+1}}=\frac{\dfrac{x-1+x+1}{x^2-1}}{\dfrac{x+1-(x-1)}{x^2-1}}=\frac{2x}{x^2-1}\div\frac{2}{x^2-1}=x.$$

By multiplying the terms of the given fraction by $(x+1)(x-1)$, we may also get directly

$$\frac{\dfrac{\cancel{(x+1)}(x-1)}{\cancel{x+1}}+\dfrac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}}{\dfrac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}-\dfrac{\cancel{(x+1)}(x-1)}{\cancel{x+1}}}=\frac{x-1+x+1}{x+1-x+1}=\frac{2x}{2}=x.$$

After a little practice, this cancellation can be done *mentally*, thus writing $\dfrac{x-1+x+1}{x+1-x+1}$ at once.

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* §§ 136, 137 may be omitted without destroying the continuity.

137. Rules for Reducing Complex Fractions.

Rule 1. *Reduce both numerator and denominator to a single fraction by performing the operations indicated. Then divide the numerator by the denominator.*

Rule 2. *Multiply both numerator and denominator by the lowest common denominator of all the small fractions.*

Then reduce the resulting fraction to its simplest form.

Rule 2 is applied in each of the following:

1. $$\frac{\frac{1}{2}+\frac{1}{3}-\frac{1}{6}}{\frac{2}{3}+\frac{5}{6}-\frac{1}{2}}=\frac{3+2-1}{4+5-3}=\frac{4}{6}=\frac{2}{3}.$$

Here we multiply both numerator and denominator of the given fraction by 6, the L. C. M. of the small denominators, 2, 3, and 6.

2. Similarly, $$\frac{\frac{1}{4}+\frac{1}{8}+\frac{1}{2}}{\frac{3}{2}-\frac{1}{4}-\frac{1}{8}}=\frac{2+1+4}{12-2-1}=\frac{7}{9}.$$

Here we multiply numerator and denominator by 8, the L. C. M. of 2, 4, and 8.

WRITTEN EXERCISES

Reduce each of the following complex fractions to its simplest form:

1. $\dfrac{1+x}{1+\frac{1}{x}}$.

2. $\dfrac{1-\frac{a}{b}}{1+a}$.

3. $\dfrac{x+\frac{x}{2}}{x-\frac{x}{2}}$.

4. $\dfrac{\frac{m+n+1}{3}}{\frac{m-n-1}{2}}$.

5. $\dfrac{4+\frac{a+b}{2}}{4-\frac{a-b}{2}}$.

6. $\dfrac{\frac{x^2-y^2}{4}}{\frac{x+y}{2}}$.

7. $\dfrac{\frac{a^3-8b^3}{27}}{a-2b}$.

8. $\dfrac{\frac{M}{D^3}}{\frac{1+d^2}{D^2}}$.

9. $\dfrac{\frac{H}{c}-\frac{hd}{cD}}{\frac{1+t}{c}}$.

10. $\dfrac{\frac{x}{x-3}-\frac{x}{x+3}}{\frac{x+2}{x-3}-\frac{x-2}{x+3}}$.

11. $\dfrac{\frac{x-3}{x+4}}{\frac{x+2}{x-4}} \div \dfrac{x-4}{x+2}$.

Simplify the following:

12. $\dfrac{\dfrac{1}{x-4}+\dfrac{1}{x-3}}{\dfrac{x}{x^2-7x+12}}$.

13. $\dfrac{1}{1+\dfrac{1}{1+x}}$.

14. $\dfrac{a}{a+\dfrac{a}{a+x}}$.

15. $\dfrac{\dfrac{2}{x-1}-\dfrac{1}{x-2}}{\dfrac{3}{x-3}-\dfrac{4}{x-1}}$.

16. $\dfrac{\dfrac{2}{x-3}+\dfrac{2x^2+2}{x-2x^3}}{\dfrac{1}{x-3}-\dfrac{1}{1-2x^2}}$.

17. $\dfrac{\dfrac{b}{a-b}+\dfrac{b}{a+b}}{\dfrac{a}{a+b}-\dfrac{a}{a-b}}$.

REVIEW QUESTIONS

1. How is a fraction defined in algebra?

2. In what ways may a fraction be changed in form without changing its value? State Principle XVII.

3. How is a fraction reduced to lowest terms?

4. How are fractions reduced to a common denominator?

5. What are the three signs of a fraction? How may these be changed without affecting the value of the fraction? Explain

$$\frac{8}{4}=-\frac{8}{-4}=-\frac{-8}{4}=\frac{-8}{-4} \text{ and } \frac{a}{b}=-\frac{a}{-b}=-\frac{-a}{b}=\frac{-a}{-b}.$$

6. State the following principles in words:

(1) $\dfrac{a}{b}=\dfrac{ma}{mb}$ (2) $\dfrac{a}{c}\pm\dfrac{b}{c}=\dfrac{a\pm b}{c}$

(3) $\dfrac{a}{b}\times\dfrac{c}{d}=\dfrac{ac}{bd}$ (4) $\dfrac{a}{b}\div\dfrac{c}{d}=\dfrac{ad}{bc}$

7.* Explain how a complex fraction may be reduced to a simple fraction in two different ways. Which is the shorter method?

CHAPTER XI

EQUATIONS INVOLVING FRACTIONS

138. **Example 1.** Solve $n+\frac{n}{2}+\frac{n}{3}=88.$ (1)

Solution. Multiplying both members of equation (1) by 6, which is a common multiple of 2 and 3, these denominators are cancelled, and we get at once

$$6n+3n+2n=528. \quad (2)$$

Uniting terms, $11n=528.$ (3)

Dividing both sides by 11, $n=48.$ (4)

The object is to multiply both members of the equation by a number that will cancel each denominator. Hence the multiplier must contain each denominator as a factor.

Evidently 12 or 18 might have been chosen for this purpose, but not 8 or 10. 6 is the smallest number which will cancel both 2 and 3, because 6 is the L. C. M. of 2 and 3.

Example 2. Solve $\frac{1}{x}+\frac{2}{x-1}=\frac{1}{x(x-1)}.$ (1)

Solution. Multiplying both members of equation (1) by the L. C. D., $x(x-1)$, we have,

$$\frac{\cancel{x}(x-1)}{\cancel{x}}+\frac{2\,x\cancel{(x-1)}}{\cancel{x-1}}=\frac{\cancel{x}\cancel{(x-1)}}{\cancel{x}\cancel{(x-1)}}. \quad (2)$$

Cancelling, we have

$$x-1+2x=1. \quad (3)$$

Hence $3x=2$ and $x=\frac{2}{3}.$

Here, as in Example 1, the members of the given equation are multiplied by an expression which cancels each denominator.

139. **Clearing of Fractions.** The process explained in the foregoing solution is called *clearing of fractions.*

As another illustration solve the equation

$$\frac{4}{x+1}-\frac{1}{x-1}=\frac{4}{(x+1)(x-1)}. \qquad (1)$$

Here the lowest common multiple is $(x+1)(x-1)$.

When we multiply $\frac{4}{x+1}$ by $(x+1)(x-1)$, the denominator $x+1$ is cancelled, thus, $\frac{4(x+1)(x-1)}{x+1}=4(x-1).$

Similarly, $-\frac{(x+1)(x-1)}{x-1}=-(x+1),$

and $\frac{4(x+1)(x-1)}{(x+1)(x-1)}=4.$

Hence, equation (1) becomes $4(x-1)-(x+1)=4.$ (2)

By *F*, $4x-4-x-1=4.$ (3)

By *F*, and *A* [5, $3x=9$ (4)

By *D* | 3, $x=3.$ (5)

Check. Substituting $x=3$ in (1) we get

$$1-\tfrac{1}{2}=\tfrac{4}{8} \text{ or } \tfrac{1}{2}=\tfrac{1}{2}$$

After a little practice the cancelling of the denominators can be done *mentally*, as we multiply by the least common denominator in clearing of fractions. For example, in this case, equation (2) may be written at once.

The foregoing examples illustrate the following

Rule. *To clear an equation of fractions, multiply both members by the L. C. D. of all the fractions involved, and then cancel the denominators.*

WRITTEN EXERCISES

Solve the following equations, indicating the principles used at each step in Examples 1 to 10:

1. $\frac{n}{2}+\frac{n}{3}=5.$

2. $n+\frac{n}{2}+\frac{n}{3}+\frac{n}{4}=25.$

3. $\frac{n}{4}+4n-\frac{5n}{3}=\frac{3n}{2}+26.$

4. $\frac{n}{2}+\frac{n}{3}-\frac{n}{4}+\frac{n}{10}=82.$

5. $7x+\frac{4x}{7}+\frac{3x}{5}+23=\frac{x}{5}+\frac{2x}{5}+5x+113.$

6. $4n+\frac{6n}{7}=\frac{3n+2}{2}+46.$

7. $12+\frac{4(9x+6)}{3}-\frac{2(3+11x)}{5}=\frac{5(4x+4)}{3}+14.$

8. $13b+7-\frac{9b+8}{7}=\frac{2b+9}{5}+38.$

9. $\frac{5a+7}{2}+\frac{2a+4}{3}=\frac{3a+9}{4}+5.$

10. $\frac{17a-5}{3}-\frac{10a+2}{4}=\frac{5a+7}{2}-5.$

11. $\frac{5x+3}{2}+\frac{5(2x+10)}{5}=2x+24.$

12. $\frac{3}{x}-\frac{1}{2}=\frac{1}{x}.$

13. $\frac{1}{3x}+\frac{1}{2x}=\frac{1}{6}.$

14. $\frac{1}{4x}+\frac{1}{3x}=\frac{5}{6x}+\frac{1}{2}.$

15. $\frac{1}{x}+\frac{2}{x-1}=\frac{8}{x(x-1)}.$

16. $\frac{4}{x+1}-\frac{3}{x-1}=\frac{-2}{x^2-1}.$

17. $\frac{1}{x-1}+\frac{1}{x+1}=\frac{4}{x^2-1}.$

18. The sum of two numbers is 12, and the first number is $\frac{1}{3}$ as great as the second. What are the numbers?

19. The smaller of two numbers is $\frac{3}{8}$ of the larger. If their sum is 66, what are the numbers?

20. Find two consecutive integers such that $\frac{5}{7}$ of the first minus $\frac{3}{11}$ of the second equals 9.

21. Find three consecutive integers such that $\frac{1}{3}$ of the sum of the first and second minus $\frac{1}{2}$ the third equals 5.

22. Find three consecutive integers such that $\frac{1}{2}$ of the first plus $\frac{1}{3}$ of the second minus $\frac{1}{10}$ of the third equals 28.

140. Special Cases. I. Sometimes it is best to *add fractions before multiplying* by the L. C. D. as in the following example:

Solve $$\frac{1}{x-2}-\frac{1}{x-1}=\frac{1}{x-4}-\frac{1}{x-3}. \quad (1)$$

Adding fractions on the right and left,

$$\frac{x-1-(x-2)}{(x-2)(x-1)}=\frac{x-3-(x-4)}{(x-4)(x-3)}.$$

Simplifying the numerators,

$$\frac{1}{(x-2)(x-1)}=\frac{1}{(x-4)(x-3)}. \quad (2)$$

Multiplying by the L. C. D. of all the fractions,

$$(x-4)(x-3)=(x-2)(x-1). \quad (3)$$

Hence, solving, $$x=2\tfrac{1}{2}. \quad (4)$$

Check by substituting $x=2\frac{1}{2}$ in equation (1).

NOTE. — If we attempt to solve this equation by first clearing of fractions, we have

$$(x-1)(x-4)(x-3)-(x-2)(x-4)(x-3)$$
$$=(x-2)(x-1)(x-3)-(x-2)(x-1)(x-4).$$

The solution of this equation is much more laborious than the solution of equation (2) above.

II. Sometimes it is best to multiply by the L. C. D. of *part of the denominators first*, and, after simplifying, multiply by the L. C. D. of the remaining denominators.

Solve $$\frac{4t-3}{16}-\frac{t-2}{4}=\frac{2t-2}{5t+2}. \quad (1)$$

Multiplying by 16, $$4t-3-4(t-2)=\frac{32t-32}{5t+2}. \quad (2)$$

Hence, $$5=\frac{32t-32}{5t+2}. \quad (3)$$

Multiplying by $5t+2$, $$25t+10=32t-32. \quad (4)$$

Hence, solving, $$t=6. \quad (5)$$

Check by substituting $t=6$ in equation (1).

In this example, try to solve by clearing of fractions completely at the outset to see whether the plan of solving here given is simpler.

WRITTEN EXERCISES

1. $\frac{3x+6}{5}-\frac{9x+3}{15}=\frac{x+7}{6x-8}+4$. *Ans.* $x=\frac{17}{19}$.

2. $\frac{7x+1}{12}-\frac{14x-22}{24}=\frac{11x+5}{8x-28}$.

3. $\frac{3x+4}{2}-\frac{12x+1}{8}=\frac{5x-1}{3x+2}$. *Ans.* $x=-7\frac{3}{5}$.

4. $\frac{7t+3}{5}-\frac{21t+9}{15}=\frac{17t-3}{3t+11}+2$. *Ans.* $t=-\frac{19}{23}$.

5. $\frac{11v-15}{10}-\frac{33v+15}{30}=\frac{5v+5}{v-5}$. *Ans.* $v=\frac{5}{7}$.

6. $\frac{x-1}{x-2}+\frac{x-2}{3-x}=\frac{x-3}{x-4}-\frac{x-4}{x-5}$. *Ans.* $x=3\frac{1}{2}$.

7. $\frac{1}{x-1}-\frac{2}{2x+1}=\frac{1}{x-2}-\frac{2}{2x-1}$. *Ans.* $x=\frac{3}{4}$.

8. $\frac{x-2}{x-3}-\frac{x-3}{x-4}=\frac{x-4}{x-5}+\frac{x-5}{6-x}$.

9. $\frac{9}{x-7}-\frac{9}{x-2}=\frac{5}{x-8}-\frac{5}{x+1}$.

10. $\frac{x+17}{x+5}-\frac{2(x+6)}{x+3}=-\frac{x-1}{x+3}$.

11. $\frac{3x-4}{x+5}-\frac{4x-1}{x+4}+\frac{x^2+44}{x^2+9x+20}=0$.

PROBLEMS LEADING TO FRACTIONAL EQUATIONS

1. What number must be subtracted from each term of the fraction $\frac{11}{17}$ so that the result shall be equal to $\frac{1}{2}$?

2. What number must be subtracted from each term of the fraction $\frac{21}{35}$ so that the result shall be equal to $\frac{2}{3}$?

3. What number must be added to each term of the fraction $\frac{1}{2}$ to obtain a fraction equal to $\frac{16}{17}$?

4. What number must be added to each of the terms in the fractions $\frac{2}{5}$ and $\frac{16}{37}$ in order to make the resulting fractions equal?

5. There are three numbers such that the second is 4 more than 9 times the first, and the third is 2 more than 6 times the first. If $\frac{1}{8}$ of the third is subtracted from $\frac{1}{7}$ of the second, the remainder is 3. Find the numbers.

6. There are three numbers such that the second is 2 more than 9 times the first and the third is 5 more than 11 times the first. The remainder when $\frac{1}{3}$ of the third is subtracted from $\frac{1}{2}$ of the second is one. Find the numbers.

7. What number must be subtracted from both the numerator and the denominator of the fraction $\frac{5}{7}$ in order to make the result equal to $\frac{1}{2}$?

8. What number must be subtracted from both numerator and denominator of the fraction $\frac{5}{3}$ in order that the fraction may be increased threefold? *Ans.* $2\frac{1}{2}$.

9. What number added to both numerator and denominator of the fraction $\frac{3}{4}$ will double the fraction?

10. Find a number of two digits in which the tens' digit is 3 greater than the units' digit, and such that if the number is divided by the sum of the digits, the quotient is 7.

11. In a number of two digits the units' digit exceeds the tens' digit by 4, and when the number is divided by the sum of its digits the quotient is 4. Find the number.

12. **Illustrative Problem.** B can do a piece of work in 8 days, and A can do it in 10 days. In how many days can they do it working together?

Since B can do the work in 8 days, in 1 day he can do $\frac{1}{8}$ of it, and since A can do it in 10 days, in 1 day he can do $\frac{1}{10}$ of it. If x is the number of days required when both work together, then in 1 day they can do $\frac{1}{x}$ of it. Hence we have the equation,

$$\frac{1}{8} + \frac{1}{10} = \frac{1}{x}.$$

13. B can do a piece of work in 12 days and A can do it in 9 days. How long will it take both working together to do it?

14. A pipe can fill a cistern in 11 hours and another in 13 hours. How long will it require both pipes to fill it?

Ans. $5\frac{2}{2}\frac{3}{4}$ hours.

15. B can do a piece of work in a days and A can do it in b days. How long will it take both together to do it?

16. A cistern can be filled by one pipe in 20 minutes and by another in 30 minutes. How long will it take to fill the cistern when both are running together?

17. A pipe can fill a cistern in 12 hours, another in 10 hours, and a third can empty it in 8 hours. How long will it require to fill the cistern when they are all running?

18. A man can do a piece of work in 18 days, another in 21 days, a third in 24 days, and a fourth in 10 days. How long will it require them when all are working together?

Ans. $4\frac{52}{617}$ days.

REVIEW QUESTIONS

1. What is meant by *clearing an equation of fractions*?

2. What principle is used in clearing an equation of fractions?

3. By what expression must both members of an equation be multiplied in order to clear the equation of fractions?

4. Is it always best to multiply *at once* by the least common multiple of all the denominators?

5. In the first example solved on page 184, state what advantage was gained by adding the fractions on each side before clearing of fractions.

6. In the second example solved on page 184, what advantage was gained by clearing of fractions *partially* at first?

CHAPTER XII

RATIO AND PROPORTION

141. Ratio. A fraction is often called a *ratio.* Thus $\frac{a}{b}$ may be read *the ratio of a to b*, and it may also be written $a : b$.

Terms of a Ratio. The numerator is called the *antecedent* of the ratio, and the denominator the *consequent.* The antecedent and consequent are called the *terms* of the ratio.

Proportion. An equation, each of whose members is a ratio, is called a *proportion.*

Thus, $\frac{a}{b} = \frac{c}{d}$ is a proportion. A proportion is usually written

$$a : b = c : d, \quad \text{or} \quad a : b :: c : d.$$

It is read *the ratio of a to b equals the ratio of c to d*, or briefly, *a is to b as c is to d.*

Means and Extremes. The four numbers a, b, c, and d, when written $\frac{a}{b} = \frac{c}{d}$, or $a : b :: c : d$, are said to be *in proportion.* Then the two end terms, a and d, are called the *extremes* of the proportion, and the two middle terms, b and c, the *means.*

HISTORICAL NOTE

Ratio and Proportion. The subject of ratio and proportion was studied fully by Euclid (300 B.C.) in connection with geometric magnitudes, but of course all his results apply equally well to numbers. Euclid, however, did not regard a ratio as a fraction and his treatment is exceedingly complicated as compared with the modern treatment in which a ratio is recognized as a fraction.

In 1631 Oughtred, an Englishman, wrote the proportion $a : b = c : d$ in the form $a : b :: c : d$. Before his time this proportion was written $a - b - c - d$. John Wallis (see opposite page) brought the symbol :: into common use and it has survived to the present time, though its meaning is exactly the same as that of =.

John Wallis (1616–1703) was an English clergyman who made contributions to mathematics, logic, and grammar. He was educated at Emmanuel College, Cambridge, and was afterward chosen fellow of Queen's College.

During the period of the conflict between King Charles I and Parliament, Wallis was an adherent of the party of the latter, and displayed surprising talent in deciphering intercepted papers and letters of the Royalists.

Wallis was the author of numerous works relating to logic, English grammar, and especially to mathematics. His work on algebra, *De Algebra Tractatus*, contains a history of the subject.

IMPORTANT PROPERTIES OF A PROPORTION

142. Transforming a Proportion. Starting each time with the proportion, $a : b :: c : d$, we deduce the following results:

Case I: $\boldsymbol{ad = bc.}$

Proof. Writing $a : b :: c : d$, in the fractional form, $\frac{a}{b} = \frac{c}{d}$, we clear of fractions and obtain $ad = bc$.

That is: *If four numbers are in proportion, the product of the means equals the product of the extremes.*

Case II: $\boldsymbol{b : a :: d : c.}$

We are to show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

Proof. From Case I, we have $bc = ad$.

Dividing both sides by ac, we get $\frac{b}{a} = \frac{d}{c}$, or $b : a :: d : c$.

This process is called taking the proportion by **inversion.**

Case III: $\boldsymbol{a : c :: b : d.}$

We are to show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

Proof. From Case I, we have $ad = bc$.

Dividing both sides by cd, we get $\frac{a}{c} = \frac{b}{d}$, or $a : c :: b : d$.

This process is called taking the proportion by **alternation.**

Case IV: $\boldsymbol{a + b : b :: c + d : d.}$

We are to show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

Proof. We have $\frac{a}{b} = \frac{c}{d}$.

Adding 1 to both sides, we get $\frac{a}{b} + 1 = \frac{c}{d} + 1$.

Hence $\frac{a+b}{b} = \frac{c+d}{d}$, or $a + b : b :: c + d : d$.

This process is called taking the proportion by **addition,** or by **composition,** as it is sometimes called.

Case V: $a - b : b :: c - d : d.$

We are to show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$

Proof. We have $\frac{a}{b} = \frac{c}{d}$.

Subtracting 1 from each side, we get $\frac{a}{b} - 1 = \frac{c}{d} - 1$.

Hence $\frac{a-b}{b} = \frac{c-d}{d}$, or $a - b : b :: c - d : d$.

This process is called taking the proportion by **subtraction,** or by **division,** as it is sometimes called.

Case VI: $a + b : a - b :: c + d : c - d.$

We are to show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

Proof. We have from Cases IV and V,

(1) $\frac{a+b}{b} = \frac{c+d}{d}$; and (2) $\frac{a-b}{b} = \frac{c-d}{d}$.

Dividing equation (1) by (2),

$$\frac{a+b}{\cancel{b}} \times \frac{\cancel{b}}{a-b} = \frac{c+d}{\cancel{d}} \times \frac{\cancel{d}}{c-d}.$$

Hence $\frac{a+b}{a-b} = \frac{c+d}{c-d}$, or $a + b : a - b :: c + d : c - d$.

This process is called taking the proportion by **addition** and **subtraction,** or by **composition** and **division.**

Case VII. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then $\frac{a+c+e}{b+d+f} = \frac{a}{b}$.

Proof. Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$; then $a = bk$, $c = dk$, $e = fk$.

Hence $a + c + e = bk + dk + fk = (b + d + f)k$,

and $$\frac{a+c+e}{b+d+f} = k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$$

That is, *The sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Mean Proportional. If $a : b :: b : x$, then b is called a *mean proportional* between a and x.

Fourth Proportional. If $a : b :: c : x$, then x is called a *fourth proportional* to a, b, and c.

WRITTEN EXERCISES

1. If $ad = bc$, show that $\frac{a}{b} = \frac{c}{d}$. *Hint.* Divide by bd.

2. If $ad = bc$, show that $\frac{a}{c} = \frac{b}{d}$.

3. If $ad = bc$, show that $\frac{d}{c} = \frac{b}{a}$.

4. If $ad = bc$, show that $\frac{d}{b} = \frac{c}{a}$.

5. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{a} = \frac{c+d}{c}$.

6. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{a} = \frac{c-d}{c}$.

7. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{a+b} = \frac{c-d}{c+d}$.

8. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{c+d} = \frac{a-b}{c-d}$.

9. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{c+d} = \frac{a}{c}$.

10. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+c}{b+d} = \frac{a}{b}$.

11. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{c-d} = \frac{a}{c}$.

12. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-c}{b-d} = \frac{a}{b}$.

13. Solve the equation $\frac{a}{b} = \frac{c}{d}$ for each letter in terms of all the others. If $a = 3$, $b = 5$, $c = 8$, find d. If $b = 7$, $c = 9$, $d = 3$, find a. If $c = 13$, $d = 2$, $a = 5$, find b. If $d = 50$, $a = 3$, $b = -7$, find c.

14. Find a fourth proportional to 3, 5, and 7; also to 9, 5, and 1; and to 3, -2, and -5.

15. Is 4 a mean proportional between 2 and 8? Why?
Is 16 a mean proportional between 4 and 32? Why?

PROBLEMS INVOLVING RATIOS

1. Which is the greater ratio, $\frac{5+3d}{5+4d}$ or $\frac{5+4d}{5+5d}$?

Hint. Reduce the fractions to a common denominator and compare numerators. (d is a positive number.)

2. Which is the greater ratio, $\frac{a+7b}{a+8b}$ or $\frac{a+9b}{a+10b}$?

3. Which is the greater ratio, $\frac{a}{b}$ or $\frac{a+c}{b+c}$, if b and c are positive, and a less than b? a equal to b? a greater than b?

4. Find two numbers in the ratio of 3 to 5 whose sum is 160.

Hint. Call the numbers x and $160 - x$.

5. Find two numbers in the ratio of 2 to 7 whose sum is -108.

6. Find two numbers in the ratio of 3 to -4 whose sum is -15.

7. What number added to each of the terms of the ratio $\frac{5}{7}$ makes it equal to $\frac{6}{7}\frac{5}{5}$?

8. What number must be added to each term of the ratio $\frac{1}{11}$ to make it equal to the ratio $\frac{3}{5}$?

9. What number added to each of the numbers 3, 5, 7, 10, will make the sums in proportion, when taken in the given order?

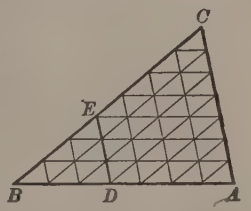
10. Two numbers are in the ratio of 2 to 3, and their sum is 60. Find the numbers.

11. What number must be subtracted from each of the numbers 7, 8, 9, and 12, so that the resulting differences shall form a proportion when taken in the given order?

SIMILAR TRIANGLES

143. Similar Triangles. Triangles are called *similar* if they have the *same shape.*

The triangles ABC and DBE in the figure are similar. *The lengths of the pairs of corresponding sides of two similar triangles form a proportion.*



That is, we may write $\frac{AB}{DB} = \frac{CB}{EB} = \frac{CA}{ED}$.

Note that AB, BC, $\cdots$ represent the **lengths** of these sides. AB and DB have been divided into 7 and 3 equal parts, respectively. Hence the ratio of these sides is $\frac{7}{3}$. In like manner, by counting the small triangles in the figure, it is found that the areas of triangles BAC and BDE are in the ratio of 49 to 9, that is, *the areas of two similar triangles are in the same ratio as the squares of corresponding sides.*

WRITTEN EXERCISES

(1) If in two similar triangles the sides of the first are 11, 13, and 16, and the side of the second which corresponds to 11 in the first is 33, find the other sides of the second triangle.

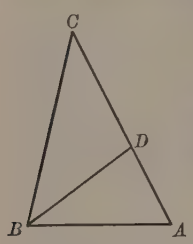
Solution. Let x represent the length of the side corresponding to the one whose length is 13. Then $\frac{x}{13} = \frac{33}{11}$, and $11\,x = 13 \cdot 33$, or $x = 39$.

(2) If the sides of a triangle are 4, 6, and 10, and one side of a similar triangle is 9, find the remaining sides of the second triangle, if the given side corresponds to the side 4.

3. Solve Example 2 if the given side in the second triangle corresponds to 6.

4. Solve Example 2 if the given side of the second triangle corresponds to 10.

5. A triangular field, one of whose sides is 20 rods, has an area of 80 square rods. Find the area of a similar field whose corresponding side is 45 rods. See § 143.



It is found in geometry that if a *line divides one angle of a triangle into two equal angles, it divides the opposite side into two parts which are in the same ratio as the other two adjacent sides of the triangle.*

That is, in the figure $\frac{AD}{DC} = \frac{AB}{BC}$.

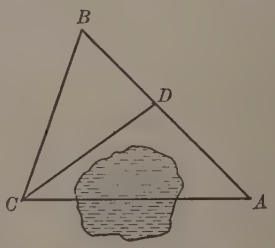
6. Knowing this fact, how many of the lines AD, DC, BC, and AB must you measure in order to find the rest of them?

7. If in the figure $AD = 6$, $DC = 9$, and $BC = 12$, find AB.

8. If in the figure $BC = 18$, $AB = 12$, and $AD = 6$, find DC.

This fact about geometry enables us in some cases to find the distance between two points without measuring it directly.

9. If in the figure CD divides the angle at C into two equal parts and if you know the lengths of AD, DB, and BC, show fully how to find the length of a straight line across the pond from C to A without measuring it directly.



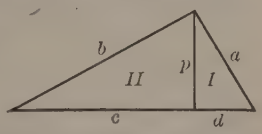
10. If $BC = 75$ rods, $BD = 50$ rods, and $AD = 60$ rods, compute AC.

11. If the sides of a triangle are a, b, c, and the corresponding sides of a similar triangle are a', b', c', show that $\frac{a+b+c}{a'+b'+c'} = \frac{a}{a'}$.

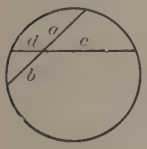
12. Two corresponding sides of two similar triangles are in the ratio 13 : 14. Show that the perimeters (sum of the sides) of the triangles are in the same ratio.

13. The perimeters of two similar triangles are in the ratio 32 : 35. Two sides of the first triangle are 8 and 12. Find two sides of the second triangle corresponding to the given sides of the first.

14. In the figure each of the triangles I and II is similar to the original triangle. From these triangles read three proportions, using the principle of § 143.



15. It follows that the triangles I and II are similar to each other. Read three proportions from this fact.



16. If in a circle two intersecting chords are drawn, as in the figure, it is known that $ab = cd$. Form a proportion from this equation. If $a = 9$, $b = 8$, $d = 6$, find c.

REVIEW QUESTIONS

1. Define *ratio, proportion, means, extremes.*

2. State in words the way in which the following proportions are derived from $\frac{a}{b} = \frac{c}{d}$.

(1) $\frac{b}{a} = \frac{d}{c}$. (2) $\frac{a}{c} = \frac{b}{d}$. (3) $\frac{a+b}{b} = \frac{c+d}{d}$.

(4) $\frac{a-b}{b} = \frac{c-d}{d}$. (5) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

3. Define *mean proportional, fourth proportional.*

4. What can you say of the corresponding sides of two similar triangles?

Draw a large figure like that in Example 14, above. Measure the sides and verify the proportions.

CHAPTER XIII

LITERAL EQUATIONS AND THEIR USES

144. Advantage in using Literal Equations. Some of the advantages of algebra over arithmetic in solving problems have been pointed out in the preceding chapters.

A further advantage is set forth in the present chapter; namely, the opportunity offered in algebra to *summarize the solution of a whole class of problems* by solving what is called a **literal equation**, thus obtaining a **formula** which may be used in solving all particular problems of this type.

For example, in arithmetic we solved many problems obtaining the **interest** when the **principal, rate**, and **time** were given. We now see that all of these can be summarized in the one literal equation, which is our *first formula for interest problems:*

$$i = prt. \tag{1}$$

Furthermore. the rules for obtaining the principal, the rate, and the time may now be derived directly from this equation by Principle VI, thus obtaining:

$$p = \frac{i}{rt}, \quad r = \frac{i}{pt}, \quad t = \frac{i}{pr}.$$

Translate each of these formulas into a rule of arithmetic.

145. Solving a Literal Equation. The process of deriving $p = \frac{i}{rt}$ from $i = prt$ is called *solving* the equation $i = prt$ for p in terms of i, r, and t, or simply *solving the equation for p.* Similarly, the process of deriving $r = \frac{i}{pt}$ from $i = prt$ is called *solving the equation for i*; and deriving $t = \frac{i}{pr}$ is called *solving the equation for t.*

In arithmetic a problem is said to be solved when a *numerical answer* is obtained which satisfies the conditions given. The solutions thus far found in algebra have, for the most part, been of this sort.

It is customary, however, to say that a problem has been solved in the *algebraic sense* when a formula is found which gives complete directions for deriving the numerical answer.

Thus, $p = \frac{i}{tr}$ is a solution for the principal since it states precisely how to find the principal in terms of interest, rate, and time.

It is thus seen that from the literal equation $i = prt$ we obtain the complete solution of every problem which calls for any one of these four numbers in terms of the other three.

In modern times machines are used extensively for computation. The algebraic solution of a literal equation *gets the problem ready for the computing machine;* that is, it gets the **formula** which the computer must use.

I. INTEREST PROBLEMS

Oral

1. If \$200 is invested at 5 %, what is the amount at the end of one year?

This problem calls for the *amount*, which is the sum of the principal and interest.

2. If \$500 is invested at 6 %, what is the amount at the end of one year?

3. If \$1000 is invested at 4 %, what is the amount at the end of one year?

4. If \$500 is invested at 5 %, what is the amount at the end of 2 years?

5. If \$1000 is invested at 6 %, what is the amount at the end of 5 years?

6. If $i = prt$, how would you represent the amount in terms of p, r, and t?

Another Interest Formula. If a = amount, then

$$a = p + prt, = p(1 + rt). \qquad (2)$$

This is our *second formula for interest problems.*

This equation may now be solved for any one of the four letters.

Thus, solving for p, we have $p = \frac{a}{1 + rt}$.

Using this equation we may find the principal when the amount, rate, and time are given.

Translated into a rule, the formula $p = \frac{a}{1 + rt}$ reads:

To find the principal, divide the amount by 1 plus the product of the rate and the time.

Written Problems

1. Find the principal if the rate is 5 %, the time 2 years, and the amount \$ 880.

Solution. $$p = \frac{a}{1 + rt} = \frac{880}{1 + .10} = \frac{880}{1.1} = \underline{800.}$$

2. Find the principal if the rate is 6 %, the time 4 years, and the amount \$ 1488.

3. Solve the equation $a = p + prt$ for t. Translate the resulting formula into words.

4. Find the time if the principal is \$ 2500, the amount \$ 3100, and the rate 4 %.

5. Find the time if the principal is \$ 5000, the amount \$ 6050, and the rate 6 %.

6. Solve the equation $a = p + prt$ for r and translate the resulting formula into words.

7. Find the rate if the principal is \$ 1800, the amount \$ 2124, and the time 4 years.

8. Find the rate if the principal is \$ 3500, the amount \$ 4340, and the time 8 years.

We have thus used the formula $a = p + prt$ to solve all possible types of problems where the amount, principal, rate, and time are involved. The points to be noticed are: (1) the great ease with which the equation may be solved for any one of its letters, thus obtaining new rules of arithmetic; (2) the convenience of solving problems by direct substitution in a formula. See the solution of Problem 1.

II. PROBLEMS INVOLVING MOTION

146. Formulas for Motion Problems. The space passed over by a moving body is called the **distance**, and the number of units of distance traversed is represented by d. The **rate** of uniform motion, that is, the number of units of space traversed in each unit of time, is called the **speed** or **rate**, and is represented by r. The number of units of time occupied is represented by t.

For example, if a train runs 40 miles per hour, in 5 hours it will run 5×40 miles.

Again, if sound travels 1080 feet per second, in 5 seconds it will travel 5×1080 feet.

In each of these examples the distance passed over is found by multiplying the rate by the time. Using the symbols d, r, and t, we have *the first formula for motion:*

$$d = rt. \qquad (1)$$

1. Solve the equation $d = rt$ for t in terms of d and r, and for r in terms of d and t. See § 145.

Translate each of these formulas into words.

It is to be understood in all problems here considered that the speed remains the same throughout the period of motion; *e.g.* sound travels just as far in any one second as in any other second of its passage.

2. If sound travels 1080 feet per second, how far does it travel in 6 seconds?

3. If a transcontinental train has an average of 35 miles per hour, how far does it travel in $2\frac{1}{2}$ days? Here we have given $r = 35$, $t = 2\frac{1}{2} \times 24$, and we are to find d.

4. A hound runs 23 yards per second and a hare 21 yards per second. If the hound starts 79 yards behind the hare, how long will it require to overtake the hare?

If t is the number of seconds required, then by formula (1) during this time the hound runs $23\ t$ yards and the hare runs $21\ t$ yards. Since the hound must run 79 yards farther than the hare, we have $23\ t = 21\ t + 79$.

The data involved in this problem may be represented conveniently in the following form:

	Rate	Time	Distance
Hound	23	t	$23\ t$
Hare	21	t	$21\ t + 79$

5. An ocean liner making 21 knots an hour leaves port when a freight boat making 8 knots an hour is already 1240 knots out. In how long a time will the liner overtake the freight?

	Rate	Time	Distance
Liner	21	t	$21\ t$
Freight boat	8	t	$8\ t + 1240$

Make a similar diagram for each of the following problems:

6. A motor boat starts 7 miles behind a sailboat and runs 13 miles per hour while the sailboat makes 6 miles per hour. How long will it require the motor boat to overtake the sailboat?

7. A freight train running 25 miles an hour is 200 miles ahead of an express train running 45 miles an hour. How long before the express will overtake the freight?

8. A bicyclist averaging 12 miles an hour is 52 miles ahead of an automobile running 20 miles an hour. How soon will the automobile overtake him?

9. A and B run a mile race. A runs 18 feet per second and B $17\frac{1}{2}$ feet per second. B has a start of 30 yards. In how many seconds will A overtake B? Which will win the race?

If in each of the examples 4 to 9 we call the rate of the faster moving object r_1 (read *r one*) and that of the slower r_2 (read *r two*), then the distance traveled by the first in the required time t is $r_1 t$, and that traveled by the second is $r_2 t$.

Then if n is the distance which the first must go in order to overtake the second, we have *the second formula for motion:*

$$r_1 t = r_2 t + n. \qquad (2)$$

The solution of (2) for t gives the time required in each problem for the first to overtake the second.

Equation (2) summarizes the solution of all problems like those from 4 to 9.

It is important that the formulas for motion problems, (1) on page 199, and (2) just derived, should be clearly understood, since they are constantly used in solving problems of this kind.

10. A fleet, making 11 knots per hour, is 1240 knots from port when a cruiser, making 19 knots per hour, starts out to overtake it. How long will it require?

Use formula (2).

11. In how many minutes does the minute hand of a clock gain 15 minute spaces on the hour hand?

Using one minute space for the unit of distance and 1 minute as the unit of time, the rates are 1 and $\frac{1}{12}$ respectively, since the hour hand goes $\frac{1}{12}$ of a minute space in 1 minute. Letting t be the number of minutes required, we have, using formula (2), $1 \cdot t = \frac{1}{12} t + 15$.

12. In how many minutes after 4 o'clock will the hour and minute hands be together? (Here the minute hand must gain 20 minute spaces.) *Ans.* $21\frac{9}{11}$ min.

13. At what time between 5 and 6 o'clock is the minute hand 15 minute spaces behind the hour hand? At what time is it 15 minute spaces ahead?

Since, at 5 o'clock, it is 25 minute spaces behind the hour hand, in the first case it must gain $25 - 15 = 10$ minute spaces, and in the second case it must gain $25 + 15 = 40$ minute spaces. Make a diagram as in the preceding problem to show both cases.

14. At what time between 9 and 10 o'clock is the minute hand of a clock 30 minute spaces behind the hour hand? At what time are they together?

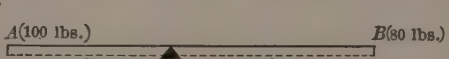
15. A fast freight leaves Chicago for New York at 8:30 A.M., averaging 32 miles per hour. At 2:30 P.M. a limited express leaves Chicago over the same road, averaging 55 miles per hour. In how many hours will the express overtake the freight?

If the express requires t hours to overtake the freight, the latter had been on the way $t + 6$ hours. Then the distance covered by the express is $55\ t$, and the distance covered by the freight is $32\ (t + 6)$. As these must be equal, we have $55\ t = 32\ (t + 6)$.

16. In a bicycle road race one rider averages $19\frac{1}{2}$ miles per hour, while another, starting 40 minutes later, averages $22\frac{1}{4}$ miles per hour. In how long a time will the latter overtake the former?

III. PROBLEMS INVOLVING THE LEVER

Two boys, A and B, play at seesaw. They find that the teeter board will balance when equal products are obtained by multiplying the weight of each by his distance from the point of support.



Thus, if B weighs 80 pounds and is 5 feet from the point of support, then A, who weighs 100 pounds, must be 4 feet from this point, since $80 \times 5 = 100 \times 4$.

The teeter board is a certain kind of lever; the point of support is called the **fulcrum**.

In each of the following problems make a diagram similar to the figure on the opposite page:

1. A and B weigh 90 and 105 pounds respectively. If A is seated 7 feet from the fulcrum, how far is B from that point?

2. Using the same weights as in the preceding problem, if B is $6\frac{1}{2}$ feet from the fulcrum, how far is A from that point?

3. A and B are 5 and 7 feet respectively from the fulcrum. If B weighs 75 pounds, how much does A weigh?

4. A and B weigh 100 and 110 pounds respectively. A places a stone on the board with him so that they balance when B is 6 feet from the fulcrum and A $5\frac{1}{2}$ feet from this point. How heavy is the stone?

Formula for Lever Problems. If the distances from the boys to the fulcrum are respectively d_1 and d_2, and their weights w_1 and w_2, then

$$d_1 w_1 = d_2 w_2.$$

This is our *formula for solving lever problems.*

This equation is a statement in the language of algebra of a very important law of nature. The law is the result of a very large number of careful experiments. It is a universal custom among scientific men, to express laws of nature, so far as possible, by means of literal equations of this sort.

If any three of the four numbers d_1, w_1, d_2, w_2, are given, the fourth may be found by means of the equation $d_1 w_1 = d_2 w_2$.

5. Solve $d_1 w_1 = d_2 w_2$ for d_1 in terms of the other three letters. Then solve for w_1.

6. Solve $d_1 w_1 = d_2 w_2$ for d_2 and also for w_2.

7. A and B are seated at the opposite ends of a 13-foot teeter board. Using the weights of problem 1, where must the fulcrum be located so that they shall balance?

If the fulcrum is the distance d from A, then it is $(13 - d)$ from B. Hence, $90\,d = 105(13 - d)$.

8. A, who weighs 75 pounds, sits 7 feet from the fulcrum. If B weighs 105 pounds, at what distance from the fulcrum should he sit in order to make a balance?

9. A and B together weigh $212\frac{1}{2}$ pounds. They balance when A is 6 feet, and B $6\frac{3}{4}$ feet, from the fulcrum. Find the weight of each.

10. A lever 9 feet long carries weights of 17 and 32 pounds at its ends. Where should the fulcrum be placed so as to make the lever balance?

11. A lever of unknown length is balanced when weights of 30 and 45 pounds are placed on it at opposite ends. Find the length of the lever, if the smaller weight is two feet farther from the fulcrum than the greater.

SUGGESTION. Let x be the distance from the greater weight to the fulcrum.

IV. MISCELLANEOUS LITERAL EQUATIONS

Solve the following for each letter in terms of the others:

1. $F = 32 + \frac{9}{5}C$.
2. $l = a + (n-1)d$.
3. $s = \frac{n}{2}(a+l)$.
4. $s = \frac{a - rl}{1 - r}$.

Solve each of the following for x:

5. $ax + 3b = cx + d$.
6. $(a-x)(b+x) = x(b-x)$.
7. $(x+a)(x+b) = (x-c)^2$.
8. $\frac{ax+b}{c} - \frac{bx+c}{d} = 1$.
9. $\frac{x}{b} + \frac{x}{a-b} = \frac{a}{a+b}$.
10. $\frac{x+1}{x-1} = \frac{a+b}{a-b}$.
11. $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = 1$.
12. $\frac{a+bx}{a+b} = \frac{c+dx}{c+d}$.
13. $\frac{a+b}{1+x} = \frac{a-b}{1-x}$.
14. $\frac{3ax}{a-b} - 2a = 5x$.
15. $\frac{ax-1}{bx} = \frac{1+bx}{ax} + 1$.

16. $\frac{x-a}{a-b}-\frac{x+a}{a+b}=\frac{2x}{a+b}$.

17. $\frac{a+x}{a}-\frac{b+x}{b}-\frac{c+x}{c}=1$.

18. $\frac{x+1}{x-1}+\frac{x-1}{x+1}=\frac{2x(x+2)}{(x+1)(x-1)}$.

19. $\frac{x+m}{x-m}+\frac{x-m}{x+m}=\frac{2x(x+1)}{(x+m)(x-m)}$.

20. $\frac{x+b}{x-a}+\frac{x-c}{x+a}=\frac{2x(x+a)}{(x+a)(x-a)}$.

21. What number must be added to each term of the fraction $\frac{3}{7}$ to obtain a fraction equal to $\frac{1}{2}$?

22. What number must be added to each term of the fraction $\frac{a}{b}$ to obtain a fraction equal to $\frac{c}{d}$?

Solve Example 21 by substituting in the formula obtained under Example 22.

23. What number must be subtracted from each term of the fraction $\frac{3}{10}$ to obtain a fraction equal to $\frac{2}{7}$?

24. What number must be subtracted from each term of the fraction $\frac{a}{b}$ to obtain a fraction equal to $\frac{c}{d}$?

Solve Example 23 by substituting in the formula obtained under Example 24.

25. What number must be added to each of the numbers 2, 5, 4, 12 so that the sums shall be in proportion when taken in the given order?

26. What number must be added to each of the numbers a, b, c, d so that the resulting sums shall be in proportion when taken in the given order?

27. What number must be subtracted from each of the numbers a, b, c, d so that the resulting remainders shall be in proportion when taken in the given order?

Compare the results in Examples 26 and 27 and explain the relation between them.

HISTORICAL NOTE

Literal Equations whose solutions lead to formulas belong to a very late stage in the development of algebra. In the earliest attempts to solve equations, the unknown was represented by some word such as *res*, the thing (see page 41). Later a single letter was used for the unknown, but known quantities were still represented by numerals, and the equations were so-called *numerical* equations. Finally, both known and unknown quantities were represented by letters, and such equations were called *literal* equations. Numerical equations lead only to *special* solutions, while literal equations lead to *general* solutions or **formulas**.

The Greeks solved special cases of general problems (see page 229), but never obtained general formulas. Vieta (see page 101) used letters, as we do, to represent both known and unknown quantities, but he always used capital letters. Harriot (see page 101), following Vieta, likewise used letters, but usually small letters instead of capitals. Newton (see page 62) was the first to let a letter stand for negative as well as positive numbers. For further uses of literal equations see page 217.

REVIEW QUESTIONS

1. State the rules which may be derived from the formula

$$v = lwh,$$

where these letters represent the volume, length, width, and height, respectively, of a rectangular solid.

2. State all the rules of interest which may be derived from the formula $i = prt$.

3. State all the rules of interest which may be derived from the formula $a = p + prt$.

4. Which problems on motion in this chapter can be solved by use of the formula

$$d = rt\,?$$

5. Which problems on motion in this chapter belong to the class whose solutions are summarized by the solutions of the equation $r_1t = r_2t + n$?

6. State fully the meaning of the equation $w_1d_1 = w_2d_2$ in connection with the lever.

CHAPTER XIV

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE

147. An Equation of the First Degree. An equation of the *first degree* in x and y is one in which neither x nor y is multiplied by itself or by the other. For instance, such an equation must not contain x^2, y^2, or xy.

E.g. $13x - 5y = 14$ is of the first degree in x and y, while $2xy - x = 5$ and $3x - 5y^2 = 13$ are not of the first degree in x and y.

Indeterminate Equations. A single equation in two unknowns is satisfied by indefinitely many pairs of values of the unknowns. Such an equation is called an *indeterminate equation.*

E.g. $x + y = 5$ is satisfied by

$$\left.\begin{matrix} x = 1 \\ y = 4 \end{matrix}\right\};\ \left.\begin{matrix} x = 2 \\ y = 3 \end{matrix}\right\};\ \left.\begin{matrix} x = 0 \\ y = 5 \end{matrix}\right\};\ \left.\begin{matrix} x = -1 \\ y = 6 \end{matrix}\right\};\ \text{etc.}$$

Simultaneous Equations. Two equations in two unknowns which are satisfied by the same pair of values of the unknowns are called *simultaneous equations.*

E.g. $x + y = 5$ and $x - y = 3$ are simultaneous, because both are satisfied by $x = 4$, $y = 1$.

Contradictory Equations. Two equations in two unknowns, which cannot be satisfied by the same pair of values of the unknowns, are called *contradictory equations.*

E.g. $x + y = 5$ and $x + y = 2$ are contradictory, since no pair of values of x and y can make their sum both 5 and 2 at the same time.

Dependent Equations. Two equations in two unknowns are said to be *dependent* if one can be derived from the other. In this case every pair of values which satisfies one will also satisfy the other.

E.g. $x + y = 5$ and $2x + 2y = 10$ are dependent, since the second may be derived from the first by multiplying both members by 2.

Independent Equations. Two equations in two unknowns are *independent* if neither can be derived from the other.

E.g. $x+y=5$ and $x-y=3$ are independent equations.

Summary.

The equations $\left\{\begin{matrix} x+y=5 \\ 2x+2y=10 \end{matrix}\right\}$ are *simultaneous* but not *independent.*

The equations $\left\{\begin{matrix} x+y=5 \\ x+y=2 \end{matrix}\right\}$ are *independent* but not *simultaneous.*

The equations $\left\{\begin{matrix} x+y=5 \\ x-y=3 \end{matrix}\right\}$ are both *simultaneous* and *independent.*

If two equations of the first degree in x and y are both **simultaneous and independent,** they have one and only one pair of values of x and y which satisfies both.

E.g. $x+y=5$ and $x-y=3$ are simultaneous and independent, and they are satisfied by $x=4$, $y=1$ and by no other pair of values of x and y.

148. Elimination. To solve two simultaneous independent equations of the first degree in x and y is to find the one pair of values of x and y which satisfies both. In this chapter such equations will be solved by the algebraic method called **elimination**. The essential step in elimination consists in combining the equations in such a way as *to get rid of one* of the unknown numbers.

ELIMINATION BY ADDITION OR SUBTRACTION

149. Illustrative Examples.

(1) Solve $\left\{\begin{matrix} x+2y=7, & (1) \\ 3x-4y=1. & (2) \end{matrix}\right.$

Multiplying both members of equation (1) by 2, we have,

$$2x+4y=14. \qquad (3)$$

Adding the members of equations, (2) and (3), $+4y$ and $-4y$ cancel.

Hence, $5x=15$, and $x=3$.

Substituting in (1), $3+2y=7$, and $y=2$.

Verify this by substituting $x=3$, $y=2$, in (1) and (2).

(2) Solve $\begin{cases} 7x+3y=4x-y+10, & (1) \\ 3x-y=4y-3x+7. & (2) \end{cases}$

Collecting the unknowns in each equation, we have,

From (1), $3x+4y=10,$ (3)

From (2), $6x-5y=7.$ (4)

Multiplying both members of equation (3) by 2, we have,

$$6x+8y=20. \quad (5)$$

Subtracting equation (4) from equation (5), we have,

$$13y=13. \quad (6)$$

By $D \mid 13$, $y=1.$

Substituting in (3), $x=2.$

Hence, the solution is $x=2, y=1.$

The process used in the solution of example (1) is called **elimination by addition**; that used in example (2) is called **elimination by subtraction.**

Rule for Elimination by Addition or Subtraction.

(*A*) *When the coefficients of one of the unknowns are numerically equal in the two equations:*

(1) *If these coefficients have opposite signs, add the equations; if they have like signs, subtract them.*

(2) *Solve the resulting equation, thus finding the value of one of the unknowns. Substitute this value in one of the given equations, thus finding the other unknown.*

(3) *Check the results by substituting in both of the given equations.*

(*B*) *When the coefficients of neither one of the unknowns are numerically equal, make one pair so by multiplying the equations by suitable numbers, and then proceed as in* (*A*).

Note. — We commonly say "add two equations" instead of using the longer expression "add the corresponding members of the equations." Similarly for subtraction. Likewise, we say "multiply the equation" and "divide the equation" instead of "multiply both members of the equation" and "divide both members of the equation."

ORAL EXERCISES

Solve the following pairs of equations by addition or subtraction. Check each solution.

1. $\begin{cases} x+y=2, \\ x-y=0. \end{cases}$
2. $\begin{cases} x+y=3, \\ x-y=1. \end{cases}$
3. $\begin{cases} x+y=4, \\ x-y=2. \end{cases}$
4. $\begin{cases} x+y=5, \\ x-y=1. \end{cases}$
5. $\begin{cases} x+y=6, \\ x-y=2. \end{cases}$
6. $\begin{cases} 2x+y=3, \\ x-y=0. \end{cases}$
7. $\begin{cases} x+2y=5, \\ x+y=4. \end{cases}$
8. $\begin{cases} 2x+y=10, \\ x+y=6. \end{cases}$
9. $\begin{cases} 3x+2y=7, \\ 3x+y=5. \end{cases}$

WRITTEN EXERCISES

Solve the following pairs of equations. Check the first six.

1. $\begin{cases} 2x+3y=22, \\ x-y=1. \end{cases}$
2. $\begin{cases} 5x-2y=21, \\ x-y=6. \end{cases}$
3. $\begin{cases} 6x+30=8y, \\ 3y+17=2-3x. \end{cases}$
4. $\begin{cases} 8x-4y=12x, \\ 4x+2y=3+4y. \end{cases}$
5. $\begin{cases} x+6y=2x-16, \\ 3x-2y=24. \end{cases}$
6. $\begin{cases} 5x+10y=-7, \\ 2x+5y=-2. \end{cases}$
7. $\begin{cases} 5x+3y=-2, \\ 3x+2y=-1. \end{cases}$
8. $\begin{cases} 3a+7b=7, \\ 5a+3b=29. \end{cases}$
9. $\begin{cases} r=3s-19, \\ s=3r-23. \end{cases}$
10. $\begin{cases} 2p=5q-16, \\ 7q=-3p+5. \end{cases}$
11. $\begin{cases} 7m=2n-3, \\ 19n=6m+89. \end{cases}$
12. $\begin{cases} 15k=10-20l, \\ 25k-30l=80. \end{cases}$
13. $\begin{cases} 6c+15d=-6, \\ 21d-8c=-74. \end{cases}$
14. $\begin{cases} 2x-3y=4, \\ 2y-3x=-21. \end{cases}$
15. $\begin{cases} u+v=27, \\ \frac{2}{3}v=19-\frac{3}{4}u. \end{cases}$
16. $\begin{cases} 7a=1+10y, \\ 16y=10a-1. \end{cases}$
17. $\begin{cases} 28x+14y=23, \\ 14x-14y=1. \end{cases}$
18. $\begin{cases} 5x+2y=x+18, \\ 2x+3y=3x+27. \end{cases}$
19. $\begin{cases} 7y-x=x-17, \\ 2y+3x=38. \end{cases}$
20. $\begin{cases} 6x+2y=-2, \\ x-4y=-35. \end{cases}$

ELIMINATION BY SUBSTITUTION

150. Illustrative Problem. Solve the equations:

$$\begin{cases} 2x + 3y = 13. & (1) \\ 5x - 6y = -8. & (2) \end{cases}$$

From (1) $3y = 13 - 2x$, or $$y = \frac{13 - 2x}{3}. \quad (3)$$

Substituting in (2), $$5x - \frac{2}{\cancel{6}}\frac{\cancel{6}(13 - 2x)}{\cancel{3}} = -8. \quad (4)$$

By F, $$5x - 26 + 4x = -8. \quad (5)$$

By I, A, $$9x = 18. \quad (6)$$

By D, $$x = 2. \quad (7)$$

From (3), $$y = \frac{13 - 2 \cdot 2}{3} = 3. \quad (8)$$

Verify this by substituting these values of x and y in (1) and (2).

The process here used is called **elimination by substitution.**

This process is convenient when no fraction remains after the substitution. This was the case in the above solution because the coefficient of y in (2) is a multiple of that in (1).

Rule for Elimination by Substitution.

(1) *Solve one equation for one unknown in terms of the other unknown.*

(2) *Substitute the expression for this unknown in the second equation.*

(3) *Solve the resulting equation, thus finding the value of one unknown.*

(4) *Substitute this value in one of the given equations, thus finding the second unknown.*

(5) *Check by substituting the results in both of the given equations.*

ORAL EXERCISES

1. If $x = 6$, what is the value of y in $2x + y = 16$?
2. If $x = 4$, what is the value of y in $x + 3y = 16$?
3. If $x = y$, what is the value of y in $x + 2y = 6$?
4. If $x = y$, what is the value of x in $3x + 2y = 15$?
5. If $x = 2y$, what is the value of y in $x + 2y = 8$?

WRITTEN EXERCISES

Solve the following by substitution:

1. $\begin{cases} x+2y=4, \\ 2x+y=5. \end{cases}$

2. $\begin{cases} 3x-y=5, \\ 5x+2y=23. \end{cases}$

3. $\begin{cases} 2x+y=3, \\ 3x-7y=30. \end{cases}$

4. $\begin{cases} 5y+x=7, \\ 5x-3y=4-2x+7. \end{cases}$

5. $\begin{cases} 5x+8y=-1, \\ 6y-x=4y-7. \end{cases}$

6. $\begin{cases} x-y=37, \\ 2x+3y=31x+13y. \end{cases}$

7. $\begin{cases} 2x-y=y+6, \\ x+2y=4y+3. \end{cases}$

8. $\begin{cases} 5x+3y=0, \\ 2x+y=1. \end{cases}$

9. $\begin{cases} 3x-2y=3, \\ 2x+3y=6x-1. \end{cases}$

10. $\begin{cases} 5x-3y=0, \\ 2x-6y=-x. \end{cases}$

WRITTEN EXERCISES

Solve by either method of elimination:

1. $\begin{cases} x+y=4, \\ x-y=10. \end{cases}$

2. $\begin{cases} x-y=-3, \\ x+4y=12. \end{cases}$

3. $\begin{cases} 2x+3y=5, \\ 7x-5y=33. \end{cases}$

4. $\begin{cases} 3x-4y=8, \\ 4x+3y=-6. \end{cases}$

5. $\begin{cases} 2x-4y=8, \\ 3x+2y=4. \end{cases}$

6. $\begin{cases} 3x-4y=8, \\ 2x+3y=11. \end{cases}$

7. $\begin{cases} 4y-2x=2, \\ 2y+5x=7. \end{cases}$

8. $\begin{cases} 3x-y=2x-1, \\ 12x+y=14. \end{cases}$

9. $\begin{cases} 2x+3y=5, \\ 6x+14y=0. \end{cases}$

10. $\begin{cases} 4x+3y=5, \\ 7x-2y=74. \end{cases}$

11. $\begin{cases} 6y+2x=14, \\ 3y+12x=18. \end{cases}$

12. $\begin{cases} 4y+9x=-5, \\ x+y=-5. \end{cases}$

13. $\begin{cases} 3y+5x=12+2x, \\ 17x-y=4y-20. \end{cases}$

14. $\begin{cases} 6+x+y=2x-1, \\ 3y+x=6y+9. \end{cases}$

15. $\begin{cases} y+5x=2x+5, \\ 2y-3x=19. \end{cases}$

16. $\begin{cases} 6x+2y=22, \\ 10x-5y=20. \end{cases}$

17. $\begin{cases} 3x-7y=16, \\ 5x+4y=11. \end{cases}$

18. $\begin{cases} 7x-4y=3, \\ 5x+8y=5. \end{cases}$

19. $\begin{cases} 12y-10x=-6, \\ 7y+x=99. \end{cases}$

20. $\begin{cases} 7x-3y=-7, \\ 5y-9x=1. \end{cases}$

21. $\begin{cases} 7x+4y=3, \\ 2x+3y=25. \end{cases}$

22. $\begin{cases} 34x+70y=4, \\ 5x-8y=-36. \end{cases}$

23. $\begin{cases} 7x+9y=8, \\ 2x-3y=19. \end{cases}$

24. $\begin{cases} 8x+4y=49, \\ 5x-8y=28. \end{cases}$

25. $\begin{cases} 4x+7y=7, \\ 5x-2y=41. \end{cases}$

26. $\begin{cases} 8x+4y=-28. \\ 3x+9y=12. \end{cases}$

SIMULTANEOUS FRACTIONAL EQUATIONS

151. Reduction to Standard Form. The equations thus far given have for the most part been written in a standard form, $ax+by=c$, in which all the terms containing x are collected, likewise those containing y, and those which contain neither x nor y. When the equations are not given in this form, they should be reduced to this form at the outset, as in the following solution:

Example. Solve $\begin{cases} \dfrac{7y-4}{5}+\dfrac{2x-3}{2}=\frac{3}{2}. & (1) \\ \dfrac{5x-2}{3}+\dfrac{2y+1}{5}=2. & (2) \end{cases}$

Solution. (1) × 10, $14y-8+10x-15=15.$ (3)

Transposing in (3), $14y+10x=38.$ (4)

(4) ÷ 2, $7y+5x=19.$ (5)

(2) × 15, $25x-10+6y+3=30.$ (6)

Transposing in (6), $25x+6y=37.$ (7)

(5) × 5, $25x+35y=95.$ (8)

(8) − (7), $29y=58.$ (9)

(9) ÷ 29, $y=2.$ (10)

Substituting in (7), $x=1.$

Check. Substitute $x=1$, $y=2$ in (1) and (2) and see that each is satisfied.

EXERCISES

After reducing each of the following pairs of equations to the standard form, solve them by means of either process of elimination:

1. $\begin{cases} \dfrac{7x-15}{3} = y, \\ 2x - y = 3. \end{cases}$

2. $\begin{cases} \dfrac{x-3}{5y} = -2, \\ x + 7y = 6. \end{cases}$

3. $\begin{cases} \dfrac{x+y}{2} + \dfrac{x-y}{2} = 10, \\ 2x - y = 16. \end{cases}$

4. $\begin{cases} \dfrac{x-y}{5} + \dfrac{x+y}{3} = 8, \\ \dfrac{2x-y}{2} + \dfrac{3y-x}{4} = 12\frac{1}{2}. \end{cases}$

5. $\begin{cases} \dfrac{7m+8}{5} - \dfrac{7n-1}{4} = -2, \\ \dfrac{2m-4}{2} + \dfrac{n-1}{3} = -\frac{1}{3}. \end{cases}$

6. $\begin{cases} \dfrac{x-3}{4} + \dfrac{y+8}{5} = 2, \\ \dfrac{x+7}{2} + \dfrac{2y-4}{7} = 5. \end{cases}$

7. $\begin{cases} \dfrac{8a-3}{9} + \dfrac{5b-2}{3} = 13, \\ \dfrac{2a+7}{5} - \dfrac{3b+10}{10} = -3\frac{4}{5}. \end{cases}$

8. $\begin{cases} \dfrac{7y-4}{5} + \dfrac{2x-3}{2} = 7, \\ \dfrac{6x-3}{5} + \dfrac{2y+1}{5} = 7. \end{cases}$

Ans. $x = 5\frac{3}{16}$, $y = 2\frac{15}{16}$.

9. $\begin{cases} \dfrac{3y+7}{2} - \dfrac{5x-7}{3} = 10, \\ \dfrac{2x-4}{3} - \dfrac{2y-1}{4} = -2\frac{1}{4}. \end{cases}$

10. $\begin{cases} \dfrac{5+3p}{7} - \dfrac{5q-2}{4} = -2, \\ 6p + 8q = 108. \end{cases}$

11. $\begin{cases} 3x - 2y = 4, \\ \dfrac{2x-1}{5} - \dfrac{7y-4}{3} = -19. \end{cases}$

12. $\begin{cases} 5x + 7y = 89\frac{1}{2}, \\ \dfrac{2x-4}{3} + \dfrac{6y-1}{5} = 13\frac{1}{3}. \end{cases}$

13. $\begin{cases} 32x - 9y = 299, \\ \dfrac{2x-5}{7} - \dfrac{3y-1}{2} = -16. \end{cases}$

14. $\begin{cases} 5x - 12y = 4, \\ \dfrac{2x-7}{3} - \dfrac{3y-4}{4} = -2. \end{cases}$

15. $\begin{cases} \dfrac{2x-y}{5} + \dfrac{3x-y}{3} = 8, \\ \dfrac{5x+y}{10} - \dfrac{9x-2y}{5} = 5. \end{cases}$

Ans. $\begin{cases} x = 1000, \\ y = 2610. \end{cases}$

152. Equations with Literal Denominators.

Solve the equations:

$$\begin{cases} \dfrac{4}{x-y}+\dfrac{6}{x+y}=\dfrac{36}{x^2-y^2}. & (1) \\ \dfrac{3}{2x-y}-\dfrac{2}{x-3y}=\dfrac{-18}{(2x-y)(x-3y)}. & (2) \end{cases}$$

Multiplying (1) by the L. C. D., x^2-y^2,

$$4(x+y)+6(x-y)=36. \quad (3)$$

By *F*, *D*, $5x-y=18. \quad (4)$

Multiplying (2) by the L. C. D., $(2x-y)(x-3y)$,

$$3(x-3y)-2(2x-y)=-18. \quad (5)$$

By *F*, *D*, $x+7y=18. \quad (6)$

Multiplying (4) by 7, $35x-7y=126. \quad (7)$

Adding (6) and (7), $36x=144. \quad (8)$

By *D*, $x=4. \quad (9)$

Substituting, $x=4$ in (6), $y=2. \quad (10)$

Check by substituting $x=4$, $y=2$ in (1) and (2).

1. $\begin{cases} \dfrac{8x+24y}{y-2x}=3, \\ \dfrac{x-7y}{x+2y+2}=17. \end{cases}$

2. $\begin{cases} \dfrac{x}{x+1}=\dfrac{2y}{2y-3}, \\ \dfrac{3}{x-1}+\dfrac{5}{y+2}=0. \end{cases}$

3. $\begin{cases} \dfrac{3x+2}{3y-5}=\dfrac{x+1}{y-1}, \\ \dfrac{3x-2}{y+1}=\dfrac{3x-1}{y-1}-\dfrac{2}{(y-1)(y+1)}. \end{cases}$

4. $\begin{cases} \dfrac{2x-1}{x+1}-\dfrac{3y-1}{y+1}=\dfrac{-xy}{(x+1)(y+1)}, \\ \dfrac{x+2}{2y-1}+\dfrac{2x-1}{y+1}=\dfrac{5xy}{(2y-1)(y+1)}. \end{cases}$

5. $\begin{cases} \dfrac{x-3}{y-2}-\dfrac{x+2}{y+1}=\dfrac{2}{(y+1)(y-2)}, \\ \dfrac{2x+1}{2y+5}-\dfrac{x+5}{y+4}=\dfrac{7}{(2y+5)(y+4)}. \end{cases}$

153. **Special Case.** In examples like the following it is best not to reduce the equations to the integral form.

Example. Solve the equations:

$$\begin{cases} \frac{2}{x}+\frac{3}{y}=2, & (1) \\ \frac{20}{x}-\frac{21}{y}=3. & (2) \end{cases}$$

Solve the equations for $\frac{1}{x}$ and $\frac{1}{y}$ instead of for x and y.

Multiplying (1) by 7, $\frac{14}{x}+\frac{21}{y}=14.$ (3)

Adding (2) and (3), $\frac{34}{x}=17.$ (4)

Hence by D, $\frac{1}{x}=\frac{1}{2}.$ (5)

Substituting $\frac{1}{x}=\frac{1}{2}$ in (1), $\frac{1}{y}=\frac{1}{3}.$ (6)

From (5) and (6) by M, $x=2,\ y=3.$ (7)

Try to solve these equations by first clearing of fractions.

WRITTEN EXERCISES

Solve the following equations:

1. $\begin{cases} \frac{1}{x}+\frac{1}{y}=4, \\ \frac{1}{x}-\frac{1}{y}=2. \end{cases}$

2. $\begin{cases} \frac{4}{x}-\frac{7}{y}=-13, \\ \frac{3}{x}+\frac{2}{y}=12. \end{cases}$

3. $\begin{cases} \frac{5}{t}-\frac{6}{v}=2, \\ \frac{17}{v}+\frac{4}{t}=67. \end{cases}$

4. $\begin{cases} \frac{3}{x}+\frac{1}{y}=21, \\ \frac{7}{x}-\frac{9}{y}=-19. \end{cases}$

5. $\begin{cases} \frac{7}{a}-\frac{1}{b}=12\frac{1}{2}, \\ \frac{3}{a}+\frac{12}{b}=24. \end{cases}$

6. $\begin{cases} \frac{3}{x}-\frac{2}{y}=-4, \\ \frac{6}{x}+\frac{11}{y}=52. \end{cases}$

7. $\begin{cases} \frac{12}{x}-\frac{10}{y}=1, \\ \frac{9}{x}+\frac{2}{y}=15. \end{cases}$

8. $\begin{cases} \frac{6}{x}+\frac{2}{y}=1, \\ \frac{1}{x}+\frac{9}{y}=1. \end{cases}$

9. $\begin{cases} \frac{1}{x}+\frac{1}{y}=20, \\ \frac{1}{x}-\frac{1}{y}=30. \end{cases}$

SIMULTANEOUS LITERAL EQUATIONS

154. Examples. 1. The sum of two numbers is 35 and their difference is 5. What are the numbers?

Let x represent one number and y the other. Form two equations and solve them.

2. The sum of two numbers is 48 and their difference is 24. What are the numbers?

3. The sum of two numbers is $41\frac{1}{2}$ and their difference is $23\frac{1}{2}$. What are the numbers?

4. The sum of two numbers is 8590 and their difference is 3480. What are the numbers?

5. If the sum of two numbers is s and their difference is d, find the numbers.

Solution. Let x represent one of the numbers and y the other.

Then, $$x + y = s, \tag{1}$$

$$x - y = d. \tag{2}$$

Solving, we get $$x = \frac{s+d}{2} = \frac{s}{2} + \frac{d}{2},$$

and $$y = \frac{s-d}{2} = \frac{s}{2} - \frac{d}{2}.$$

Translated into words these results are:

One of two numbers is equal to half their sum plus half their difference, and the other is equal to half their sum minus half their difference.

6. Test this general solution by applying it to the following:

$\begin{cases} s = 48, \\ d = 24; \end{cases}$ $\begin{cases} s = 8590, \\ d = 348; \end{cases}$ $\begin{cases} s = 40, \\ d = 52; \end{cases}$ $\begin{cases} s = 38, \\ d = 50. \end{cases}$

7. Apply the general solution to the following:

$\begin{cases} s = \frac{3}{4}, \\ d = \frac{1}{4}; \end{cases}$ $\begin{cases} s = 3\frac{1}{5}, \\ d = 1\frac{1}{5}; \end{cases}$ $\begin{cases} s = 3\frac{1}{2}, \\ d = 1\frac{1}{4}; \end{cases}$ $\begin{cases} s = 5\frac{2}{3}, \\ d = 1\frac{2}{3}. \end{cases}$

This shows again how the solution of literal equations leads to *formulas*. See pages 196 to 206.

WRITTEN EXERCISES

In the following x and y are the unknowns. Solve for them in terms of the other letters.

1. $\begin{cases} ax + by = 1, \\ cx - dy = 1. \end{cases}$

2. $\begin{cases} 2x - 3y = c - d, \\ 3x - 2y = c + d. \end{cases}$

3. $\begin{cases} mx + 2ny = k, \\ 3 + 2mx = +ny. \end{cases}$

4. $\begin{cases} x + y = a, \\ \frac{x}{a} - \frac{y}{b} = 1. \end{cases}$

5. $\begin{cases} \frac{a}{x} + \frac{b}{y} = 1, \\ \frac{c}{x} + \frac{d}{y} = 1. \end{cases}$

6. $\begin{cases} \frac{1}{x} + \frac{1}{y} = a, \\ \frac{1}{x} - \frac{1}{y} = b. \end{cases}$

7. $\begin{cases} \frac{a}{x} + \frac{a}{y} = m, \\ \frac{b}{x} - \frac{b}{y} = n. \end{cases}$

8. $\begin{cases} \frac{a}{x} + \frac{b}{y} = r, \\ \frac{c}{x} + \frac{d}{y} = s. \end{cases}$

PROBLEMS LEADING TO FRACTIONAL EQUATIONS

In solving the following problems use two equations if two unknowns are involved.

1. Find two numbers whose sum is 51, such that if the greater is divided by their difference, the quotient is $3\frac{1}{3}$.

2. Find two numbers whose sum is 91, such that if the greater is divided by their difference, the quotient is 7.

3. There are two numbers whose sum is s, such that if the greater is divided by their difference, the quotient is q. Find an expression in terms of s and q representing each number. Solve Examples 1 and 2 by substituting in this formula.

4. There are two numbers whose difference is 153. If their sum is divided by the smaller, the quotient is equal to $\frac{47}{15}$. Find the numbers.

5. There are two numbers whose difference is d. If their sum is divided by the smaller, the quotient is q. Find the numbers. Solve Example 4 by substituting in this formula.

6. Find two numbers whose difference is 320, such that the greater divided by their sum is $\frac{3}{4}$.

7. Find two numbers whose difference is 60, such that the greater divided by their sum is $\frac{7}{8}$.

8. Find two numbers whose difference is $\frac{1}{2}$, such that the greater divided by their sum is $\frac{5}{8}$.

9. Find two numbers whose difference is d, such that the greater divided by their sum is $\frac{c}{b}$.

10. A number has two digits whose sum is s. If the number is divided by the difference between the digits, the quotient is q. Find the number, the tens' digit being the larger.

11. There is a number composed of two digits whose sum is 11. If the number is divided by the difference between the digits, the quotient is $16\frac{2}{3}$. Find the number, the tens' digit being the larger.

155. **Many problems** may be solved in two ways, namely, by using one unknown, or by using two unknowns.

Example. Find two numbers whose sum is 20, such that when one of them is subtracted from twice the other, the remainder is 16.

(a) *Using one unknown.* Let x represent one number. Then $20 - x$ is the other number, and the equation is $2x - (20 - x) = 16$.

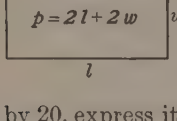
(b) *Using two unknowns.* Let x and y represent the two numbers. Then,

$$\begin{cases} x + y = 20, \\ 2x - y = 16. \end{cases}$$

The translation of problems into equations is usually easier when more than one unknown is permitted. This is due to the fact that in this case each of the given relations between the numbers is put down as a *separate equation*.

PROBLEMS INVOLVING TWO UNKNOWNS

1. If w and l are the width and length of a rectangle, express its perimeter in terms of w and l. Also its area.



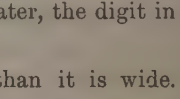
2. If the width of the rectangle in the preceding is increased by 10 and its length by 20, express its new perimeter and also its new area in terms of w and l.

3. If x and y represent the ages of a father and son respectively, represent the sum of their ages 5 years ago in terms of x and y.

4. If a number consisting of two digits is increased by 15 by changing the order of its digits, which is greater, the digit in tens' or in units' place?

5. If a number consisting of two digits is decreased by changing the order of its digits, which is greater, the digit in tens' or in units' place?

6. A rectangular field is 32 rods longer than it is wide. The length of the fence around it is 308 rods. Find the dimensions of the field.



7. Find two numbers such that 7 times the first plus 4 times the second equals 37, while 3 times the first plus 9 times the second equals 45.

8. A certain sum of money was invested at 5% interest and another sum at 6%, the two investments yielding \$980 per annum. If the first sum had been invested at 6% and the second at 5%, the annual income would be \$1000. Find each sum invested.

Suggestion. Let x and y represent the sums invested.

Then
$$\begin{cases} \frac{5}{100}x+\frac{6}{100}y=980. \\ \frac{6}{100}x+\frac{5}{100}y=1000. \end{cases}$$

9. The combined distance from the sun to Jupiter and from the sun to Saturn is 1369 million miles. Saturn is 403 million miles farther from the sun than Jupiter. Find the distance from the sun to each planet.

10. Find two numbers such that 7 times the first plus 9 times the second equals 116, and 8 times the first minus 4 times the second equals 4.

11. The sum of two numbers is 108. 8 times one of the numbers is 9 greater than the other number. Find the numbers.

12. Two investments of $24,000 and $16,000 respectively yield a combined income of $840. The rate of interest on the larger investment is 1% greater than that on the other. Find the two rates of interest.

Suggestion. $\begin{cases} x = y + 1, \\ 24{,}000 \times \frac{x}{100} + 16{,}000 \times \frac{y}{100} = 840. \end{cases}$

13. A father is twice as old as his son. Twenty years ago the father was six times as old as his son. How old is each now?

14. If the length of a rectangle is increased by 3 feet and its width decreased by 1 foot, its area is increased by 3 square feet. If the length is increased by 4 feet and the width decreased by 2 feet, the area is decreased by 3 square feet. What are the dimensions of the rectangle?

Note that if w and l are the original width and length of the rectangle, the term lw will cancel out of both equations.

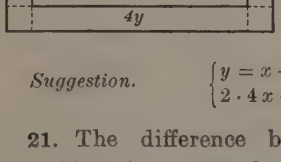
15. A steamer on the Mississippi makes 6 miles per hour going against the current and $19\frac{1}{2}$ miles per hour going with the current. What is the rate of the current and at what rate can the steamer go in still water?

16. One number is three times another If 10 is added to each of the numbers, their quotient is $\frac{5}{11}$. Find the numbers.

17. A starts at 8 A.M. for a walk in the country. At 10 A.M. B starts on horseback to overtake A, which he does at noon. If the rate of B had been one mile per hour less, he would have overtaken A at 1 P.M. At what rate does each travel?

18. A camping party sends a messenger with mail to the nearest post office at 5 A.M. At 8 A.M. another messenger is sent out to overtake the first, which he does in $2\frac{1}{4}$ hours. If the second messenger travels 5 miles per hour faster than the first, what is the rate of each?

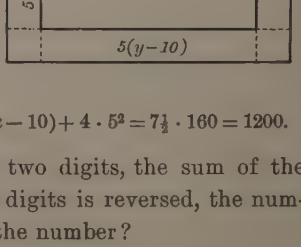
19. There are two numbers such that 3 times the greater is 18 times their difference, and 4 times the smaller is 4 less than twice the sum of the two. What are the numbers?



20. A picture is 3 inches longer than it is wide. The frame, which is 4 inches wide, has an area of 360 square inches. What are the dimensions of the picture?

Suggestion. $\begin{cases} y = x + 3. \\ 2 \cdot 4x + 2 \cdot 4y + 4 \cdot 4^2 = 360. \end{cases}$

21. The difference between two sides of a rectangular wheat field is 30 rods. A farmer cuts a strip 5 rods wide around the field, and finds the area of this strip to be $7\frac{1}{2}$ acres. What are the dimensions of the field?



Suggestion. $\begin{cases} y = x + 30. \\ 2 \cdot 5(y - 10) + 2 \cdot 5(x - 10) + 4 \cdot 5^2 = 7\frac{1}{2} \cdot 160 = 1200. \end{cases}$

22. In a number consisting of two digits, the sum of the digits is 10. If the order of the digits is reversed, the number is decreased by 54. What is the number?

23. The sum of the length and width of a certain field is 260 rods. If 20 rods are added to the length and 10 rods to the width, the area will be increased by 3800 square rods. What are the dimensions of the field?

24. In a number consisting of two digits the sum of the digits is 12. If the order of the digits is reversed, the number is increased by 36. What is the number?

25. A bird attempting to fly against the wind is blown backward at the rate of $7\frac{1}{2}$ miles per hour. Flying with the wind when it is $\frac{1}{4}$ as strong, the bird makes 48 miles an hour. Find the rate of the wind and the rate at which the bird can fly in calm weather.

26. There is a number whose two digits differ by 2. If the digit in units' place is multiplied by 3 and the digit in tens' place is multiplied by 2, the number is increased by 44. Find the number, the tens' digit being the larger.

27. In a number consisting of two digits the units' digit is equal to twice their difference. If the order of the digits is reversed, the number is increased by 18. Find the number.

28. If the length of a rectangle is doubled and 8 inches added to the width, the area of the resulting rectangle is 180 square inches greater than twice the original area. If the length and width of the rectangle differ by 10, what are its dimensions?

29. There is a number consisting of three digits, those in tens' and units' places being the same. The digit in hundreds' place is 4 times that in units' place. If the order of the digits is reversed, the number is decreased by 594. What is the number?

30. A man rowing against a tidal current drifts back $2\frac{1}{2}$ miles per hour. Rowing with this current, he can make $14\frac{1}{2}$ miles per hour. How fast does he row in still water and how swift is the current?

31. Flying against a wind a bird makes 28 miles per hour, and flying with a wind whose velocity is $2\frac{2}{5}$ times as great, the bird makes 46 miles per hour. What is the velocity of the wind and at what rate does the bird fly in calm weather?

32. A freight train leaves Chicago for St. Paul at 11 A.M. At 3 and 5 P.M. respectively of the same day two passenger trains leave Chicago over the same road. The first overtakes the freight at 7 P.M. the same day, and the other, which runs 10 miles per hour slower, at 3 A.M. the next day. What is the speed of each?

33. Two boys, A and B, trying to determine their respective weights, find that they balance on a teeter board when B is 6 feet and A is 5 feet from the fulcrum. If B places a 30-pound weight on the board beside him, they balance when B is 4 feet and A is 5 feet from the fulcrum. How heavy is each boy?

34. \$ 10,000 and \$ 8000 are invested at different rates of interest, yielding together an annual income of \$ 820. If the first investment were \$ 12,000 and the second \$ 6000, the yearly income would be \$ 840. Find the rates of interest.

SIMULTANEOUS EQUATIONS IN THREE UNKNOWNS

156. **An equation of the first degree in x, y, and z** is one which contains these letters in such a way that no one of them is multiplied by itself or by any other one. For instance, such an equation must not contain such terms as x^2, y^2, z^2, xy, xz, xyz, etc.

E.g. $2x + 3y - z = 5$ is of the first degree in x, y, and z.

157. **Systems of Equations.** Two or more equations involving the same unknowns form, when taken together, a *system of equations*. A system of three equations in three unknowns may be *simultaneous* or *contradictory*, *independent* or *dependent*, in the same sense as explained in § 147 for two equations in two unknowns.

Illustrative Problem. Three men were discussing their ages and found that the sum of their ages was 90 years. If the age of the first were doubled and that of the second trebled, the aggregate of the three ages would then be 170. If the ages of the second and third were each doubled, the sum of the three would be 160. Find the age of each.

Solution. Let x, y, and z represent the number of years in their ages in the order named.

Then, $$x + y + z = 90, \quad (1)$$

$$2x + 3y + z = 170, \quad (2)$$

and $$x + 2y + 2z = 160. \quad (3)$$

If we subtract equation (1) from (2), we obtain a new equation from which z is eliminated.

That is, $$x + 2y = 80. \quad (4)$$

Again, multiplying equation (2) by 2 and subtracting (3),

$$3x + 4y = 180. \quad (5)$$

Equations (4) and (5) involve the two unknowns x and y. Solving these by eliminating y, we find $x = 20$. (6)

Substituting $x = 20$ in (4), $y = 30$. (7)

Substituting x and y in (1), $z = 40$. (8)

Check by showing that the values of x, y, and z satisfy the original equations and the conditions of the problem.

The values of x, y, and z as thus found constitute the **solution of the given system** of equations.

Evidently x could have been eliminated first, using equations (1) and (2), and then (1) and (3), giving a new set of two equations in y and z. Let the student find the solution in this manner.

Also find the solution by first eliminating y, using (1) and (2), and then using (2), (3), getting two equations in x and z, from which the values of x and z can be found.

It will be found that the solutions are the same, no matter in what order the equations are combined. This indicates that *a system of three independent and simultaneous equations of the first degree in three unknowns has one and only one solution.*

As in the case of two equations, each should be first reduced to a standard form in which all the terms containing a given unknown are collected and united and all fractions removed.

EXERCISES

Solve the following systems, and check the results:

1. $\begin{cases} 2x-y+z=18, \\ x-2y+3z=10, \\ 3x+y-4z=20. \end{cases}$

2. $\begin{cases} 5x-3y+z=15, \\ x-3y-z=-3, \\ 2x-y+z=8. \end{cases}$

3. $\begin{cases} 4x+2y+z=13, \\ x-y+z=4, \\ x+2y-z=1. \end{cases}$

4. $\begin{cases} 6x+4y-4z=-4, \\ 4x-2y+8z=0, \\ x+y+z=4. \end{cases}$

5. $\begin{cases} x+2y+3z=5, \\ 4x-3y-z=5, \\ x+y+z=2. \end{cases}$

6. $\begin{cases} 2x-8y+3z=2, \\ x-4y+5z=1, \\ 3x-10y-z=5. \end{cases}$

7. $\begin{cases} x+y+z=1, \\ x+3y+2z=8, \\ 2x+8y-3z=15. \end{cases}$

8. $\begin{cases} 2x-3y+z=5, \\ 3x+2y-z=5, \\ x+y+z=3. \end{cases}$

9. $\begin{cases} x+y+z=6, \\ 3x-2y-z=13, \\ 2x-y+3z=26. \end{cases}$

10. $\begin{cases} x+y+z=6, \\ 4x-y-z=-1, \\ 2x+y-3z=-6. \end{cases}$

11. $\begin{cases} 2x-3y-4z=17, \\ 4x-4y+2z=-10, \\ 7x+7y+5z=17. \end{cases}$

12. $\begin{cases} x+y+z=0, \\ 5x+3y+4z=-1, \\ 2x-7y+6z=21. \end{cases}$

13. $\begin{cases} x+2y-z=2, \\ 2x-y+z=3, \\ x+2y+z=8. \end{cases}$

14. $\begin{cases} 2x-y-z=6, \\ 2x-2y+z=10, \\ x+y-3z=-2. \end{cases}$

15. $\begin{cases} x-y-z=1, \\ 2x+3y+z=20, \\ x-2y+z=0. \end{cases}$

16. $\begin{cases} 5x-2y+3z=5, \\ -2x+y-z=-1, \\ -x-y+2z=4. \end{cases}$

17. $\begin{cases} 4a-3b+2c=8, \\ a+b-4c=-16, \\ 7a-4b+c=4. \end{cases}$

18. $\begin{cases} 5m-4n+r=8, \\ 3m+n-3r=0, \\ 2m-4n+6r=28. \end{cases}$

19. $\begin{cases} 9x-4y-z=-4, \\ 2x+5y-6z=-12, \\ -x+2y+4z=30. \end{cases}$

20. $\begin{cases} 4x+7y+3z=-3, \\ x-3y+2z=16, \\ 5x+2y-4z=1. \end{cases}$

PROBLEMS INVOLVING THREE UNKNOWNS

158. Illustrative Problem. A broker invested a total of $15,000 in the street railway bonds of three cities, the first investment yielding 3 %, the second 3½ %, and the third 4 %, thus securing an income of $535 per year. If the second investment was one half the sum of the other two, what was the amount of each?

Solution. Suppose x dollars were invested at 3 %, y dollars at 3½ %, and z dollars at 4 %.

Then,
$$\left\{\begin{aligned} x + y + z &= 15000, && (1)\\ .03\,x + .035\,y + .04\,z &= 535, && (2)\end{aligned}\right.$$
and
$$x + z = 2\,y. \qquad (3)$$

From equation (3), $x - 2\,y + z = 0.$ (4)

Subtracting (4) from (1), $3\,y = 15000.$ (5)

Hence, $y = 5000.$ (6)

From (1), by M, $.035\,x + .035\,y + .035\,z = 525.$ (7)

Subtracting (7) from (2), $-.005\,x + .005\,z = 10.$ (8)

Dividing (8) by .005, $-x + z = 2000.$ (9)

Substituting (6) in (4), $x + z = 10000.$ (10)

Adding (9) and (10), $2\,z = 12000.$ (11)

$z = 6000.$ (12)

Substituting (6) and (12) in (1), $x = 4000.$ (13)

Hence, $4000, $5000, and $6000 were the sums invested.

WRITTEN PROBLEMS

Solve the following problems, using three unknowns:

1. The sum of three angles, A, B, and C, of a triangle is 180 degrees. $\frac{1}{3}$ of $A + \frac{1}{4}$ of $B + \frac{1}{5}$ of C is 48 degrees, while $\frac{1}{6}$ of $A + \frac{1}{8}$ of $B + \frac{1}{4}$ of C is 30 degrees. How many degrees in each angle?

2. The combined weight of 1 cubic foot each of compact limestone, granite, and marble is 535 pounds. 1 cubic foot of limestone, 2 of granite, and 3 of marble weigh together 1041 pounds, while 1 cubic foot of limestone and 1 of granite together weigh 195 pounds more than 1 cubic foot of marble. Find the weight per cubic foot of each kind of stone.

3. A number is composed of 3 digits whose sum is 7. If the digits in tens' and hundreds' places are interchanged, the number is increased by 180; and if the order of the digits is reversed, the number is decreased by 99. What is the number?

4. The sum of the angles A, B, and C of a triangle is 180 degrees. If B is subtracted from C, the remainder is $\frac{1}{3}$ of A, and when C is subtracted from twice A, the remainder is 4 times B. How many degrees in each angle?

5. The sum of three numbers, a, b, and c, is 35. Twice a is 5 less than the sum of b and c, and twice c is 4 more than the sum of a and b. What are the numbers?

6. If x is the number of seconds in the Eastern intercollegiate record for a mile run, y the number in the Western record, and z the number in the world's record, then

$$\begin{cases} x + y + z = 768.95, \\ -x + 2y + z = 518.95, \\ 2x - y + z = 502.75. \end{cases}$$

7. If x is the number of seconds in the Eastern intercollegiate record for a half mile run, y the number in the Western intercollegiate record, and z the number in the world's record, then

$$\begin{cases} 2x + 3y + z = 692.9, \\ 3x + 2y + 2z = 804.6, \\ 2x - y + z = 226.5. \end{cases}$$

8. If x is the number of seconds in the world's mile trotting record in 1806, y is the number of seconds in the world's record in 1885, and z is the number of seconds in the world's record in 1911, then

$$\begin{cases} x + y + z = 426.25, \\ 2x + 4y + 6z = 1584, \\ -x + y + 2z = 186.75. \end{cases}$$

9. Diophantus of Alexandria (see page 41) gives the following problem: "Find three numbers such that the sum of each pair is a given number."

It is interesting to note that Diophantus states his problem in words in its general form, but he solves it for a special case; viz., for $x + y = 20$, $y + z = 30$, $z + x = 40$. The Greeks did not use letters to represent numbers in general. Hence they had no formulas such as we now have.

Solve this special case.

10. Solve the preceding problem when the given numbers are a, b, c. That is, solve the system $x + y = a$; $y + z = b$; and $z + x = c$.

REVIEW QUESTIONS

1. Why is a single equation in two unknowns called *indeterminate?*

2. When are two such equations called *simultaneous?* When *independent?* When *contradictory?*

3. May two equations be simultaneous without being independent? May they be independent but not simultaneous?

4. Give the proper description to each of the following systems of equations:

(1) $\begin{cases} x - y = 2 \\ x + y = 10 \end{cases}$ (2) $\begin{cases} 2x + y = 7 \\ 6x + 3y = 21 \end{cases}$ (3) $\begin{cases} x + 2y = 6 \\ 2x + 4y = 10 \end{cases}$

5. Describe *elimination* by the process of *addition* or *subtraction;* also by the process of *substitution.* Under what conditions is one or the other of these methods preferable?

6. Describe the solution of a system of three linear equations in three unknowns. Is it immaterial which of the three variables is eliminated first?

7. Can you find a definite solution for two equations such as $4x - 3y - z = 5$ and $x + y + z = 2$?

Eliminate z from these equations. What is the nature of the resulting equation? (See § 147.)

CHAPTER XV

GRAPHIC REPRESENTATION*

159. Graphic Representation of Statistics. A graphic representation of the temperatures recorded on a certain day is shown on the next page. The readings were as follows:

3 P.M.	29°	9 P.M.	21°	3 A.M.	12°	9 A.M.	12°
4 P.M.	29°	10 P.M.	20°	4 A.M.	11°	10 A.M.	13°
5 P.M.	28°	11 P.M.	17°	5 A.M.	10°	11 A.M.	16°
6 P.M.	26°	12 M'T.	16°	6 A.M.	10°	12 Noon	17°
7 P.M.	24°	1 A.M.	14°	7 A.M.	10°	1 P.M.	18°
8 P.M.	22°	2 A.M.	12°	8 A.M.	10°	2 P.M.	20°

In the graph each heavy dot represents the temperature at a certain hour. The distance of the dot from the heavy vertical line indicates the hour of the day counted from noon, and its distance above the heavy horizontal line indicates the thermometer reading at that hour. The lines joining these dots complete the picture representing the gradual changes of temperature from hour to hour.

Graphs of this kind are used in commercial houses to represent variations of sales, fluctuations of prices, etc. They are used by the historian to represent changes in population, fluctuations in the amount of mineral productions, etc. In algebra they are used in solving problems and in explaining many difficult processes. In the succeeding exercises cross-ruled paper is essential.

Make a graphic representation of the tables of data on the opposite page:

In each case the number to be represented by one space on the cross-ruled paper should be chosen so as to make the graph go conveniently on a sheet. Thus in Example 1 let one small horizontal space represent two years and one vertical space a million of population; and in Example 3 let one horizontal space represent one year and one large vertical space one hundred thousand of population.

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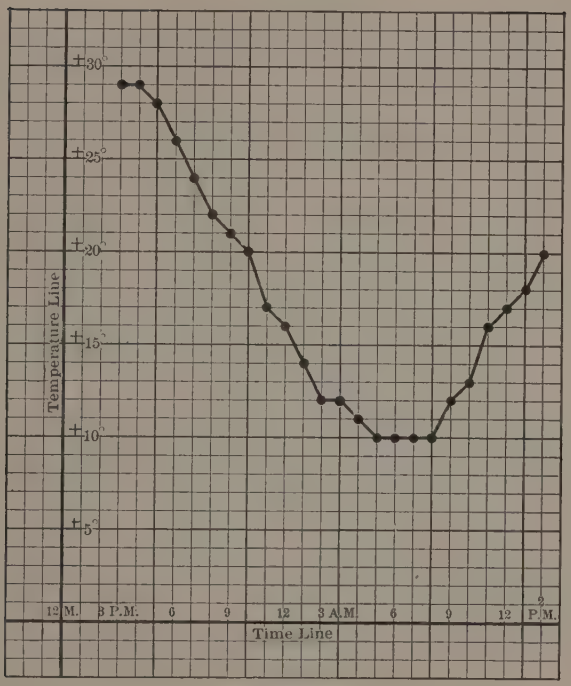
* Chapter XV may be omitted without destroying the continuity.



EXERCISES

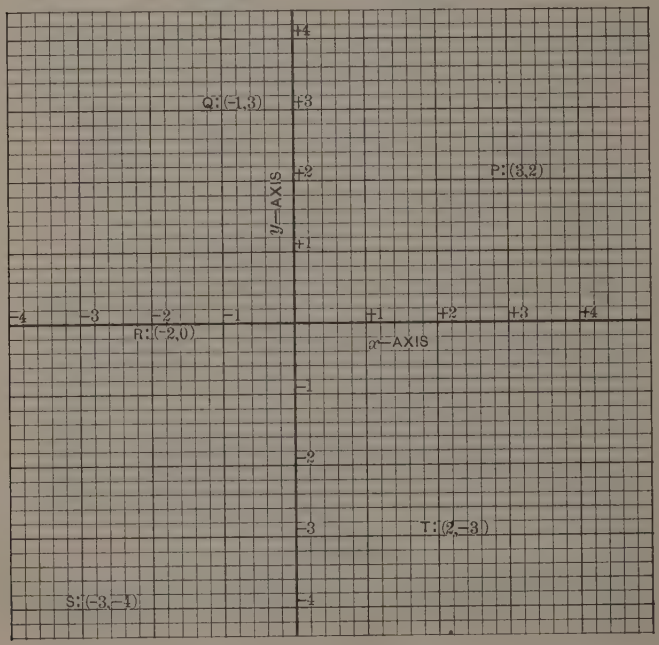
1. The population of the United States, 1800 to 1910:

1800 . . 4.3 (millions)	1840 . . 17.1	1880 . . 50.2
1810 . . 7.2	1850 . . 23.2	1890 . . 62.6
1820 . . 9.6	1860 . . 31.4	1900 . . 76.3
1830 . . 12.9	1870 . . 38.6	1910 . . 92.0

2. The population of the boroughs now constituting Greater New York City:

1800 . . 79 (thousands)	1840 . . 391	1880 . . 1912
1810 . . 119	1850 . . 696	1890 . . 2507
1820 . . 152	1860 . . 1175	1900 . . 3427
1830 . . 242	1870 . . 1478	1910 . . 4767

160. Axes. In the graphs thus far constructed two lines at right angles to each other have been used as reference lines. These lines are called **axes.** The location of a point in the plane of such a pair of axes is completely described by giving its distance and direction from each of the axes.



This scheme of locating points by two reference lines is already familiar to the pupil from geography, where cities are located by latitude and longitude; that is, by degrees north or south of the equator and east or west of the meridian of Greenwich.

The direction to the right of the vertical axis is denoted by a positive sign, and to the left, by a negative sign; while direction upward from the horizontal axis is also called positive, and downward, negative.

The horizontal line is usually called the ***x*-axis** and the vertical line the ***y*-axis.**

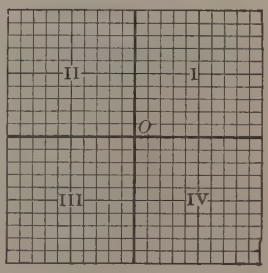
Abscissa. The perpendicular distance of any point from the y-axis is called the *abscissa* of the point.

Ordinate. The perpendicular distance of any point from the x-axis is called its *ordinate.*

Coördinates. The abscissa and ordinate of a point are together called its *coördinates.*

E.g. the abscissa of point P in the opposite figure is 3 and its ordinate 2, or we may say the coördinates of P are 3 and 2, and indicate it thus, $P:(3, 2)$, writing the abscissa first. In like manner for the other points we write $Q:(-1, 3)$, $R:(-2, 0)$, $S:(-3, -4)$, and $T:(2, -3)$.

We see that in this manner every point in the plane corresponds to a pair of numbers, and that every pair of numbers corresponds to a point.



Quadrants. The two axes divide the plane into four parts called *quadrants.* These are numbered I, II, III, IV, in counter-clockwise order, as shown in the figure.

Origin. The intersection of the two axes is called the *origin of coördinates.*

Examples. 1. Plot the point $(4, -3)$.

Solution. Since the abscissa is positive, we measure 4 units from the *y*-axis to the *right.*

Since the ordinate is negative, we measure 3 units *down* from the x-axis.

Hence, the required point is in the fourth quadrant.

The student should use squared paper and plot the point.

2. Plot the point $(-2, -3)$.

Solution. Since the abscissa is negative, we measure 2 units from the *y*-axis to the *left.*

Since the ordinate is negative, we measure 3 units *down* from the x-axis.

Hence, the point is in the third quadrant.

EXERCISES

1. With any convenient scale, locate the following points: $(2, 6)$, $(-3, 5)$, $(0, 1)$, $(1, 0)$, $(0, 0)$, $(0, -1)$, $(0, -5)$, $(-5, 0)$, $(2\frac{1}{2}, 5\frac{1}{3})$, $(-4, -8)$, $(3, -10)$, $(-10, 3)$.

2. Locate the following series of points and then see if a straight line can be drawn through them: $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 3)$, $(4, 4)$, $(-1, -1)$, $(-2, -2)$, $(-3, -3)$.

3. Locate the following and connect them by a line: $(1, 0)$, $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(1, -2)$, $(1, -3)$, $(1, -4)$, $(1, -5)$. Name other points in this line.

4. Draw the line every one of whose points has its horizontal distance -2; also draw the line every one of whose points has its vertical distance $+3$.

5. Locate the following points and see if a straight line can be passed through them: $(1, 0)$, $(0, 1)$, $(2, -1)$, $(3, -2)$, $(4, -3)$, $(-1, +2)$, $(-2, 3)$, $(-3, 4)$, $(-4, 5)$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{4}, \frac{3}{4})$, $(\frac{2}{3}, \frac{1}{3})$. Can you name other points on this line?

Points on a Straight Line. In some of the preceding exercises, a series of points has been found to *lie on a straight line*, as in Examples 2, 3, and 5. Evidently this could not happen unless the points were located according to some definite scheme or law.

In Example 5 it is easy to see what the law is; namely, the sum of the abscissa and ordinate is 1 for each point.

Thus, for the first point the sum is $1 + 0 = 1$; for the second point it is $0 + 1 = 1$; for the third, $2 + (-1) = 1$, etc.

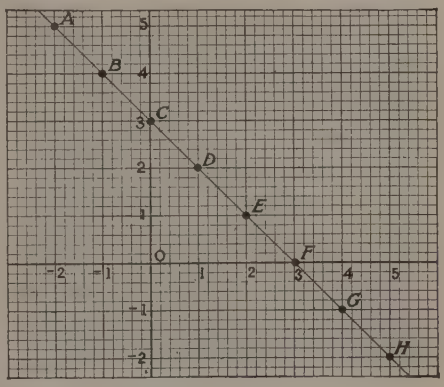
For the last point the sum is $\frac{2}{3} + \frac{1}{3} = 1$, for the next to the last it is $\frac{1}{2} + \frac{1}{2} = 1$, etc.

Hence, in Example 5, if we let x stand for any one of the abscissas, and y for its corresponding ordinate, we have $x + y = 1$ as the law by which all these points are located.

In Example 2, the law is $x = y$, and in Example 3 it is $x = 1$, whatever y may be. In Example 4, the laws are $x = -2$ for the first, and $y = 3$ for the second.

GRAPHIC REPRESENTATION OF EQUATIONS

161. Not only may statistics be represented by graphs, but there is also a way to make a graphic representation of an equation.



Example. Make a graph of the equation $x + y = 3$.

Solution. Writing the equation in the form $y = 3 - x$, we see that for every value which we give to x, we can find a corresponding value of y so as to satisfy the equation $y = 3 - x$. For instance,

if $x = 0$, $y = 3 - 0 = 3$;

if $x = 2$, $y = 3 - 2 = 1$;

if $x = 3$, $y = 3 - 3 = 0$, etc.

In this way we may construct the following table of values of x and y, and we may extend this in both directions as far as we like.

If $x =$	-2	-1	0	1	2	3	4	5 etc.,
then $y =$	5	4	3	2	1	0	-1	-2 etc.

If we call each value of x the abscissa of a point and the corresponding value of y its ordinate, this table gives the following points:

A	B	C	D	E	F	G	H
$(-2, 5)$	$(-1, 4)$	$(0, 3)$	$(1, 2)$	$(2, 1)$	$(3, 0)$	$(4, -1)$	$(5, -2)$.

If we plot these points, as in the figure, we see that they all lie on the straight line AH.

Furthermore, it can be shown that all the points whose abscissas and ordinates are values of x and y which satisfy this equation lie on this straight line AH.

Hence the line AH is called the *graph of the equation* $x + y = 3$.

162. Graph of a Linear Equation. If an equation is of the first degree in x and y, all the points whose coördinates are values of x and y which satisfy this equation lie on a straight line. This straight line is called *the graph of the equation.*

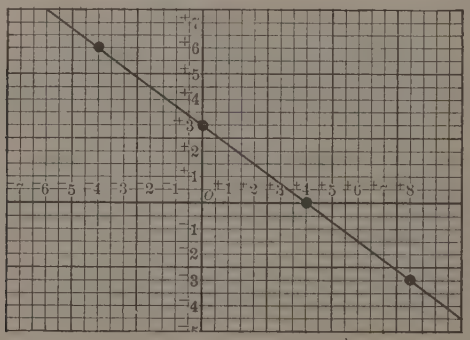
An equation which has a straight line for its graph is called a **linear equation.**

To graph a linear equation it is necessary to locate only two of its points and to draw a straight line through them. If this line is extended indefinitely in both directions, it will pass through all the points whose coördinates satisfy this equation, and through no other points.

E.g. In graphing the equation $x - y = 5$, we choose $x = 0$ and find $y = -5$, and choose $y = 0$ and find $x = 5$ and plot the points $(0, -5)$ and $(5, 0)$. The line through these points is the one required.

Example. Graph the equation $3x + 4y = 12$.

Solution. We see that when $x = 0$, $4y = 12$, and $y = 3$.
Also when $y = 0$, $3x = 12$, and $x = 4$.



Hence two required points are (0, 3) and (4, 0).

Plotting these points, we draw the straight line through them as shown in the figure.

Check. Plot the following points, whose coördinates satisfy the equation, and show that they lie on the graph.

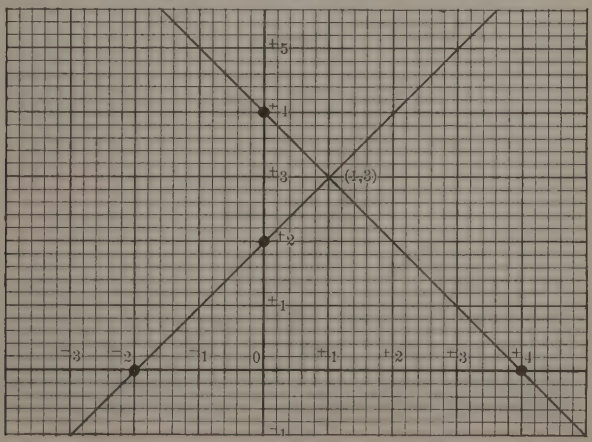
$$\begin{cases} x = 8 \\ y = -3 \end{cases} \quad \begin{cases} x = 12 \\ y = -6 \end{cases} \quad \begin{cases} x = -4 \\ y = 6 \end{cases} \quad \begin{cases} x = -8 \\ y = 9 \end{cases}$$

EXERCISES

Construct the graph for each of the following equations:

1. $3x+2y=1$.
2. $5x-3y=-3$.
3. $7x+10y=2$.
4. $x+2y=0$.
5. $5-2y=6x$.
6. $3x+5y=-15$.
7. $2x-y=0$.
8. $3x-4y=7$.
9. $3x-4y=-7$.
10. $3x-4y=-12$.
11. $7y=9x-63$.
12. $x=5y+3$.

163. **Graphic Solution.** Graph on the same axes $x+y=4$ and $y-x=2$, and thus solve this pair of equations.



Solution. The two graphs are found to intersect in the point (1, 3). Since the point lies on both lines, its coördinates should satisfy both equations, as indeed they do. Since these lines have only one point in common, there is no other pair of numbers which, when substituted for the unknowns x and y, can satisfy both equations.

Check. Solve these equations by elimination, and see if you get these same values for x and y.

Hence, $x=1$, $y=3$ is the **solution of this pair of equations.**

Since $x=1$, $y=3$ represents a point whose abscissa is 1 and whose ordinate is 3, we may write this solution as we would indicate a point; namely, (1, 3).

164. Independent and Simultaneous Equations. Since the graphs of the preceding equations are distinct, the equations are properly called *independent.* Since the graphs intersect, the coördinates of the common point satisfy both equations *simultaneously.*

165. The Graph shows that there is only one Solution. Since two straight lines intersect in but one point, it follows that *two linear equations which are independent and simultaneous have one and only one solution.* Two such equations may be solved by *finding the coördinates of the point where their graphs meet.*

EXERCISES

Graph the following and thus solve each pair of equations:

1. $\begin{cases} 2x-3y=25, \\ x+y=5. \end{cases}$

2. $\begin{cases} 5x+6y=7, \\ 2x-y=-4. \end{cases}$

3. $\begin{cases} 5x+3y=8, \\ 2x-y=-10. \end{cases}$

4. $\begin{cases} 6x+8y=16, \\ 2x-3y=11. \end{cases}$

5. $\begin{cases} 3x-4y=1, \\ 2x-7y=5. \end{cases}$

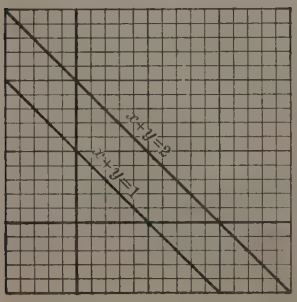
6. $\begin{cases} y+3x=7, \\ 2y+x=-6. \end{cases}$

7. $\begin{cases} x-2y=2, \\ 2x-y=-2. \end{cases}$

8. $\begin{cases} 5x-7y=21, \\ x-4y=-1. \end{cases}$

9. $\begin{cases} 5x+2y=8, \\ 2x-3y=-12. \end{cases}$

166. Dependent Equations. If we attempt to plot two equations which are not independent, such as $x+y=1$ and $2x+2y=2$, the graphs will be found to **coincide.** Such equations are properly called **dependent,** since any point which lies on the graph of one also lies on the graph of the other.



167. Contradictory Equations. If we attempt to plot two equations which are not simultaneous, such as $x+y=1$ and $x+y=2$, the graphs will be found to be parallel and hence they have no point in common. Such equations are properly called **contradictory,** since there is no point which can lie on both graphs.

HISTORICAL NOTE

Graphic Representation of Equations. The representation of equations by means of lines is due to René Descartes (1596–1650). (See next page.) By the time of Descartes the work of Vieta, Harriot, and others had given to algebra the modern form, thus perfecting it as a general instrument of research. The first important use to which it was put was the application to geometry made by Descartes. This must be regarded as one of the greatest contributions of all time to mathematics.

Not only is it possible to graph linear equations as we have done here, but it is found that many important curves of different kinds are graphs of equations of higher degrees. The points where two such curves meet furnish the solutions of the equations of which they are the graphs; and conversely, the algebraic solutions of such equations tell the points where the graphs intersect. This enables us to use the operations of algebra in solving a large range of problems pertaining to lines and curves.

The graphic method is much used by engineers and others in the solution of practical problems.

REVIEW QUESTIONS

1. How may a point in a plane be located by reference to two fixed lines? What are these lines called? What names are given to the distances from the point to the fixed lines?

2. Draw a pair of axes in a plane and locate the following points: $(5, 0)$, $(-2, 0)$, $(0, 3)$, $(0, -1)$, $(0, 0)$.

3. How many pairs of numbers can be found which satisfy the equation $x - 2y = 6$? State five such pairs and plot the corresponding points. How are these points situated with respect to each other? What can you say of all points corresponding to pairs of numbers which satisfy this equation? What is meant by the graph of an equation?

4. How many pairs of numbers will simultaneously satisfy the two equations $3x + 2y = 7$ and $x + y = 3$? Show by means of a graph that your answer is correct.

5. If negative numbers could not be used, how would the graph of $x + y = 3$ be limited? The graph of $x - y = 3$?

CHAPTER XVI

SQUARE ROOTS AND RADICALS

168. Square Root. A **square root** of a number is one of its two equal factors. See § 96.

Thus 3 is a square root of 9, since $3 \cdot 3 = 9$. Similarly $a + b$ is a square root of $a^2 + 2ab + b^2$.

It should be noted that every square has *two* square roots which are numerically equal with opposite signs.

E.g. -3 is a square root of 9 as well as $+3$, since $(-3) \cdot (-3) = 9$.

Radical Sign. The positive square root of a number is indicated by the *radical sign* $\sqrt{\ }$ alone or preceded by the sign $+$. The negative square root is indicated by the radical sign preceded by the sign $-$.

E.g. $+\sqrt{9}$ or $\sqrt{9} = +3$ and not -3, and $-\sqrt{9} = -3$, and not $+3$.

The square root of any number is at once evident if we can resolve it into two equal groups of factors.

E.g.

$$\sqrt{576} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3} = \sqrt{(2^3 \cdot 3)(2^3 \cdot 3)} = \sqrt{24 \cdot 24} = 24.$$

ORAL EXERCISES

Find the following indicated square roots:

1. $\sqrt{4}$.	6. $-\sqrt{49}$.	11. $\sqrt{196}$.	16. $-\sqrt{625}$.
2. $\sqrt{9}$.	7. $\sqrt{81}$.	12. $-\sqrt{256}$.	17. $-\sqrt{900}$.
3. $-\sqrt{16}$.	8. $\sqrt{121}$.	13. $-\sqrt{144}$.	18. $\sqrt{10000}$.
4. $\sqrt{25}$.	9. $-\sqrt{169}$.	14. $\sqrt{400}$.	19. $-\sqrt{64}$.
5. $\sqrt{36}$.	10. $\sqrt{225}$.	15. $\sqrt{289}$.	20. $-\sqrt{1600}$.

Dividing Exponents by 2. *We see that the square root of an even power is obtained by dividing its exponent by 2.*

Thus, we have $\sqrt{a^4} = a^2$, which is $a^{4\div 2}$.
Similarly, $\sqrt{a^6} = a^3$, which is $a^{6\div 2}$.
And $\sqrt{a^8} = a^4$, which is $a^{8\div 2}$.

ORAL EXERCISES

Find the following indicated square roots:

1. $\sqrt{2^4}$.	11. $\sqrt{3^{4a}}$.	21. $\sqrt{(a+b)^4}$.
2. $\sqrt{5^{12}}$.	12. $\sqrt{5^{6ab}}$.	22. $\sqrt{(a+b)^8}$.
3. $-\sqrt{7^8}$.	13. $\sqrt{a^{4n}}$.	23. $-\sqrt{(a+b)^{2n}}$.
4. $\sqrt{a^{12}}$.	14. $\sqrt{c^{6m}}$.	24. $\sqrt{(a-b)^6}$.
5. $-\sqrt{3^{14}}$.	15. $\sqrt{x^{12a}}$.	25. $-\sqrt{(a-b)^{4n}}$.
6. $\sqrt{a^{24}}$.	16. $-\sqrt{x^{4a}}$.	26. $\sqrt{(a-b)^{10r}}$.
7. $-\sqrt{3^8}$.	17. $\sqrt{y^{6b}}$.	27. $-\sqrt{(a^2-b)^{8c}}$.
8. $\sqrt{x^{2n}}$.	18. $-\sqrt{x^{8c}}$.	28. $-\sqrt{(a-b^2)^{2n}}$.
9. $\sqrt{a^{4n}}$.	19. $\sqrt{m^{16a}}$.	29. $\sqrt{(a-b+c)^4}$.
10. $\sqrt{4^{2a}}$.	20. $\sqrt{y^{10x}}$.	30. $\sqrt{(a-b+c)^{8n}}$.

169. The Square Root of a Product. The square root of the product of several factors, each of which is a square, may be found by *taking the square root of each factor separately,* as in the following examples:

(1) $\sqrt{4 \cdot 16 \cdot 25} = \sqrt{4} \cdot \sqrt{16} \cdot \sqrt{25} = 2 \cdot 4 \cdot 5 = 40.$

This is true since $4 \cdot 16 \cdot 25$ can be written as the product of two groups of equal factors $(2 \cdot 4 \cdot 5)(2 \cdot 4 \cdot 5)$. Hence, one of these $(2 \cdot 4 \cdot 5)$ is the square root of $4 \cdot 16 \cdot 25$.

(2) $\sqrt{25\,a^4b^6} = \sqrt{5^2} \cdot \sqrt{a^4} \cdot \sqrt{b^6} = 5\,a^2b^3.$

This is true since $5^2a^4b^6$ can be written as the product of two groups of equal factors $(5\,a^2b^3)(5\,a^2b^3)$. Hence, one of these $(5\,a^2b^3)$ is the square root of $5^2\,a^4b^6$.

From these examples we see that *the square root of a product may be obtained by dividing the exponent of each factor by* 2.

ORAL EXERCISES

Find the following indicated square roots:

1. $-\sqrt{2^2 \cdot 3^2}$.
2. $\sqrt{81 \cdot 121}$.
3. $\sqrt{49 \cdot 25 \cdot 169}$.
4. $-\sqrt{8^2 \cdot 5^2 \cdot 3^2}$.
5. $\sqrt{5^4 \cdot 3^2 \cdot 4^4}$.
6. $\sqrt{25 \cdot 36}$.
7. $-\sqrt{3^{12} \cdot 5^{14}}$.
8. $-\sqrt{2^{22} \cdot 3^{12}}$.
9. $\sqrt{16\, a^2b^2c^2}$.
10. $\sqrt{64\, a^4x^4}$.
11. $-\sqrt{4^4\, a^2b^4}$.
12. $\sqrt{3^2\, x^2y^2}$.
13. $\sqrt{9\, x^4y^{12}}$.
14. $-\sqrt{121\, a^2x^4}$.
15. $-\sqrt{7^4\, a^4b^2}$.

Notice that $\sqrt{9+16}$ is not equal to $\sqrt{9}+\sqrt{16}$.

The preceding exercises illustrate

Principle XVIII

170. **Rule.** *The square root of a product is obtained by finding the square root of each factor separately and then taking the product of these roots. That is,*

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}.$$

HISTORICAL NOTE

Radicals. — The essential elements in the theory of radicals were developed long before the present notation came into use. Alkarismi, the author of the first Arabian algebra (see page 32), states in substance that $a\sqrt{b} = \sqrt{a^2b}$, $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ (Principle XVIII).

Nicolas Chuquet, in a French book published in 1484, gives the earliest known use of the radical sign, though the Hindus had used a similar symbol.

Christoff Rudolff, a German writer, used the radical sign in 1525 but he indicated cube root by $\sqrt{\ }\sqrt{\ }\sqrt{\ }$ and fourth root by $\sqrt{\ }\sqrt{\ }$.

Wallis (see page 101) first used fractional exponents to indicate roots.

ORAL EXERCISES

Find the following square roots:

1. $-\sqrt{4\, a^6b^8}$.
2. $-\sqrt{3^2x^{12}y^{14}}$.
3. $\sqrt{5^2 \cdot 3^{22}t^{24}}$.
4. $\sqrt{121\, x^4y^{12}}$.
5. $-\sqrt{64\, a^2b^4}$.
6. $-\sqrt{10^4\, a^4b^4}$.
7. $\sqrt{5^4\, m^{14}}$.
8. $\sqrt{5^4 \cdot 3^8 \cdot 7^2}$.
9. $\sqrt{3^{14} \cdot 7^{12}\, a^4}$.
10. $-\sqrt{25\, a^2b^4c^{12}}$.
11. $\sqrt{81\, x^4y^8c^{10}}$.
12. $\sqrt{49\, a^6y^{10}z^{14}}$.

THE SQUARE ROOT OF A POLYNOMIAL

171. Relations between a Square and its Square Root. If we square $a + b$ we get $a^2 + 2\,ab + b^2$. To find the square root of $a^2 + 2\,ab + b^2$ we try to see how $a + b$ can be derived from $a^2 + 2\,ab + b^2$.

Step (1) We see that a can be found from a^2 by taking its square root.

Step (2) We see that b can be found from $2\,ab$ by dividing it by twice the term already found, a.

The steps in the work are arranged as follows:

The given square =	$a^2 + 2\,ab + b^2$	$\lfloor a + b =$ the square root.	
The square of a =	a^2		
Trial divisor $= 2\,a$		$2\,ab + b^2$	= first remainder.
Complete divisor $= 2\,a + b$		$2\,ab + b^2$	$= b(2\,a + b)$.
		0	

Explanation. Having found a, as indicated in step (1) above, we subtract its square from $a^2 + 2\,ab + b^2$.

The remainder begins with $2\,ab$, which we use to find b, as in step (2) above.

That is, we multiply a by 2, and divide $2\,ab$ by this product, $2\,a$, which we call the *trial divisor*, thus getting b, the second term of the root.

Now, the remainder, $2\,ab + b^2$, may be written $(2\,a + b)b$. Hence, if we add b to $2\,a$ (calling $2\,a + b$ the *complete divisor*), and multiply this by b, the product, $2\,ab + b^2$, is the rest of the square.

Thus, in finding the terms, a and b, of the root, we really build up the whole of the square and subtract it piece by piece. The remainder, zero, indicates that the square root is exact.

Example. Find the square root of $36x^2 - 84\,xy + 49y^2$.

Given square =	$36\,x^2 - 84\,xy + 49\,y^2$	$\lfloor 6\,x - 7\,y =$ the root.	
Square of $6\,x$ =	$36\,x^2$		
Trial divisor $= 2 \cdot 6\,x$		$-84\,xy + 49\,y^2$.	
Complete divisor $= 12\,x - 7\,y$		$-84\,xy + 49\,y^2$	$= -7\,y(12\,x - 7\,y)$.
		0	

Explanation. The first term of the root is $6\,x$ since $\sqrt{36\,x^2} = 6\,x$. The product of twice $6\,x$ and the second term of the root is $-84\,xy$. Hence, the second term of the root is $-84\,xy \div 12\,x = -7\,y$. The complete divisor is then $12\,x - 7\,y$, and the rest of the square is $-7\,y(12\,x - 7\,y) = -84\,xy + 49\,y^2$.

EXERCISES

Find the square root of each of the following:

1. $16x^4 - 64x^2y^2 + 64y^4$.
2. $64x^2 - 32x + 4$.
3. $25 + 49x^2 - 70x$.
4. $a^{10} + 6a^5b + 9b^2$.
5. $36x^2 - 84x + 49$.
6. $121 - 44x^4 + 4x^8$.
7. $9a^2 - 30ab^2 + 25b^4$.
8. $81a^2 - 216a + 144$.
9. $4a^2b^4 - 44ab^2 + 121$.
10. $81 - 270y + 225y^2$.

172. **Rule for Finding the Square Root of a Polynomial.** From the squares

$$(a+b)^2 = a^2 + 2ab + b^2,$$

$$(a+b+c)^2 = [(a+b)+c]^2 = (a+b)^2 + 2(a+b)c + c^2,$$

$$(a+b+c+d)^2 = [(a+b+c)+d]$$
$$= (a+b+c)^2 + 2(a+b+c)d + d^2, \text{ etc.},$$

we see that

(1) If to a^2 we add $2ab + b^2 = (2a+b)b$, we get $(a+b)^2$;

(2) If to $(a+b)^2$ we add $2(a+b)c + c^2$
$= [2(a+b)+c]c$, we get $(a+b+c)^2$;

(3) If to $(a+b+c)^2$ we add $2(a+b+c)d + d^2$
$= [2(a+b+c)+d]d$, we get $(a+b+c+d)^2$, etc.

Hence, in squaring a polynomial:

For every new ***term*** *added to the root there is a new* ***part*** *added to the power. This new part consists of twice the sum of the* ***preceding*** *terms of the root plus the* ***last*** *term of the root, all multiplied by the last term of the root.*

This is expressed by the formula:

$$(a+b+c+d)^2$$
$$= a^2 + (2a+b)b + [2(a+b)+c]c + [2(a+b+c)+d]d.$$

WRITTEN EXERCISES

Write the following squares in the above form:

1. $(x+y+z)^2$.
2. $(x+y+z+v)^2$.
3. $(x+2y+3z+4v)^2$.
4. $(2a+b+3c+d)^2$.
5. $(b+4c+2d+e)^2$.
6. $(x+2a+3b+c)^2$.

Example 1. Find the square root of

$$9x^4 - 12x^3 + 28x^2 - 16x + 16.$$

Solution.

		$a + b + c$
Square root	=	$3x^2 - 2x + 4$
Given square	=	$9x^4 - 12x^3 + 28x^2 - 16x + 16$
$a^2 = (3x^2)^2 =$		$9x^4$
$2a = 2 \cdot 3x^2 = 6x^2$		$-12x^3 + 28x^2 - 16x + 16$
$(2a + b)b = (6x^2 - 2x)(-2x) =$		$-12x^3 + 4x^2$
$2(a + b) = 6x^2 - 4x$		$24x^2 - 16x + 16$
$[2(a + b) + c]c = (6x^2 - 4x + 4) \cdot 4 =$		$24x^2 - 16x + 16$
		0

Explanation. The first term of the root is $\sqrt{9x^4} = 3x^2$. This is subtracted from the square.

The second term of the root is $-12x^3 \div (2 \cdot 3x^2) = -2x$.

The second part of the square is $(6x^2 - 2x)(-2x) = -12x^3 + 4x^2$, corresponding to $(2a + b)b$. This is now subtracted.

The third term of the root is $24x^2 \div (2 \cdot 3x^2) = 4$.

The third part of the square is $(6x^2 - 4x + 4) \cdot 4 = 24x^3 - 16x + 16$, corresponding to $[2(a + b) + c]c$. This is now subtracted.

Thus at each step a new term of the root is found by dividing the first term of the remainder by twice the first term of the root; and then a new part of the power is built up and subtracted.

Since the final remainder is zero, the square root is exact.

Example 2. Find the square root of

$$16x^6 - 24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25.$$

Solution.

		$a + b + c + d$
Square root		$4x^3 - 3x^2 + 2x - 5$
Given square		$16x^6 - 24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25$
$a^2 = (4x^3)^2 =$		$16x^6$
		$-24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25$
$(2a + b)b =$		$-24x^5 + 9x^4$
		$16x^4 - 52x^3 + 34x^2 - 20x + 25$
$(2(a + b) + c)c =$		$16x^4 - 12x^3 + 4x^2$
		$-40x^3 + 30x^2 - 20x + 25$
$[2(a + b + c) + d]d =$		$-40x^3 + 30x^2 - 20x + 25$

Explanation. In this solution only the successive parts of the formula are written down. Let the student give the explanation in full.

From the preceding examples we have the following

Rule: (1) *Arrange the polynomial according to ascending or descending powers of some letter.*

(2) *Find the square root of the first term, and write it as the first term of the root.*

(3) *Subtract the square of this first term of the root.*

(4) *Divide the first term of the remainder by twice the first term of the root, and write the quotient as the second term of the root.*

(5) *Add this second term of the root to twice the first term, and multiply the sum by the second term. This product is the second part of the square and is to be subtracted.*

(6) *Now use the sum of the first two terms of the root to find the third term, just as the first term was used to find the second; and continue in this manner till all the terms of the root are found.*

EXERCISES

Find the square root of each of the following:

1. $x^4+2x^3+3x^2+2x+1$.
2. $1-2a+3a^2-2a^3+a^4$.
3. $1+2b-b^2-2b^3+b^4$.
4. $a^4+4a^3+6a^2+4a+1$.
5. $c^4-4c^3+6c^2-4c+1$.
6. $x^4-2x^3+5x^2-4x+4$
7. $a^4+4a^3b+6a^2b^2+4ab^3+b^4$.
8. $x^4-4x^3y+6x^2y^2-4xy^3+y^4$.
9. $a^4+53a^2+14a^3+28a+4$.
10. $a^6+6a^5+15a^4+20a^3+15a^2+6a+1$.
11. $a^6-6a^5+15a^4-20a^3+15a^2-6a+1$.
12. $4x^6-12x^5+13x^4-14x^3+13x^2-4x+4$.
13. $16a^6+24a^5+25a^4+20a^3+10a^2+4a+1$.
14. $x^6y^6+2x^5y^5+3x^4y^4+4x^3y^3+3x^2y^2+2xy+1$.
15. $1+2x+3x^2+4x^3+5x^4+4x^5+3x^6+2x^7+x^8$.

THE SQUARE ROOT OF AN ARITHMETIC NUMBER

173. Rule for Finding the First Term of the Square Root of an Integral Number. Since $1^2 = 1$ and $9^2 = 81$, the square of a number of one figure contains either *one* or *two* figures.

Since $10^2 = 100$ and $99^2 = 9801$, the square of a number of two figures contains either *three* or *four* figures.

Similarly, the square of a number of three figures contains either *five* or six *figures*, and so on.

Rule. Hence, to find the first figure in the root

(1) *Separate the number into groups of two figures each, counting from units' place toward the left. The last group may contain only one figure.*

(2) *Take the square root of the largest square in the left hand group. This is the first figure of the root, and there are as many figures in the root as there are groups in the number.*

Examples. 1. To find the first figure and the number of figures in the square root of 450,769 we write it thus 45 07 69.

Since there are three groups of two figures each, the square root contains three figures, and hence it starts with the hundreds' figure.

Since 36 is the largest square in 45, the first figure in the root is $\sqrt{36} = 6$.

2. Similarly, the square root of 6,762,436, written 6 76 24 36, contains four figures of which the first one is thousands' figure.

Since 4 is the largest square in 6, the first figure of the root is $\sqrt{4} = 2$.

ORAL EXERCISES

Give the first figure in the square root of each of the following, and state whether it stands in units', tens' or hundreds' place:

1. 8947.	5. 90,401.	9. 7347.	13. 107.
2. 6205.	6. 63,401.	10. 73,470.	14. 4091.
3. 19,140.	7. 1428.	11. 14,051.	15. 10,007.
4. 72,048.	8. 194,670.	12. 140,051.	16. 100,007.

The square root of an arithmetic number may be found by the process just used for polynomials, if we remember that a number like 637 is really a polynomial, namely $600 + 30 + 7$.

Illustrative Example 1. Find the square root of 405769.

Solution.

	Square	Square Root
		$a + b + c$
	40 57 69	$\lfloor 600 + 30 + 7 = 637$
$a^2 = 600^2 =$	36 00 00	
$2a = 1200$	4 57 69	
$b = 30$		
$2a + b = 1230$	3 69 00	$= (2a + b)b$
$2(a + b) = 1260$	88 69	
$c = 7$		
$2(a + b) + c = 1267$	88 69	$= [2(a + b) + c]c$
	0	

Explanation. The first figure in the root is the square root of the largest square in the left hand group, and since there are three groups, the root starts with 600, which corresponds to a of the formula (§ 172).

Subtracting the square of 600 we have 4 57 69.

The trial divisor is $2a = 1200$ and when 4 57 69 is divided by 1200, the largest number of tens in the quotient is 3. Hence 30 corresponds to b of the formula.

The complete divisor is $2a + b = 1230$, and this multiplied by b gives $(2a + b)b = 36900$, which is the second part of the square. Subtracting 36900, the remainder is 88 69.

The next trial divisor is $2(a + b) = 1260$ and $8869 \div 1260$ gives 7 as the largest number of units. This is c of the formula. Then

$$2(a + b) + c = 1267,$$

and this multiplied by 7 gives $[2(a + b) + c]c = 88\ 69$.

The remainder is now zero, and hence the square root is

$$600 + 30 + 7 = 637.$$

In case a square consists of a **whole number and a decimal part** the figures in the *integral part* of the square root are found exactly as in Example 1 above. To find the *decimal part* of the root, we proceed as in the next illustrative example.

ORAL EXERCISES

Give the first figure in the square root of each of the following, and tell in which place it stands:

1. 12.645.	**4.** 941.61.	**7.** 49.29.
2. 1.2645.	**5.** 94.16.	**8.** 4.929.
3. 126.45.	**6.** 9.416	**9.** 492.9.

Illustrative Example 2. Find the square root of 67.7329.

Solution.

	Square	Square Root	
		$a+b+c$	
	67.73 29	$8+.2+.03=8.23$	
$a^2=8^2=$	64		
$2a=16$	3.73 29		
$b=.2$			
$2a+b=16.2$	3.24		$=(2a+b)b$
$2(a+b)=16.4$	.49 29		
$c=.03$			
$2(a+b)+c=16.43$	.49 29		$=[2(a+b)+c]c$
	0		

Explanation. There is only one group of two figures to the left of the decimal point. Hence the first figure of the root is units' figure. *Since the square of a decimal contains twice as many decimal places as the number itself, there will be* **one** *decimal figure in the root for every* **two** *in the square.*

In getting the second figure of the root, the trial divisor is $2a=16$. The quotient is $b=.2$ since $.2\times 16=3.2$. The quotient could not be .3 since $.3\times 16=4.8$.

Similarly, in getting the third figure, we divide .4929 by 16.4 and the quotient is .03 since $.03\times 16.4=.492$.

Illustrative Example 3. Find the square root of 9.1204

Solution.

	Square	Square Root	
		$a+b$	
	9.1204	$3+.02=3.02$	
$a^2=3^2=$	9		
$2a=6$	.12 04		
$b=.02$			
$2a+b=6.02$	.12 04		$=(2a+b)b$
	0		

Explanation. Since there is only one group to the left of the decimal point, the first figure of the root is in units' place.

In this case in dividing .1204 by 6, the quotient is .02 since $.02\times 6=12$; that is, there is a zero in tenths' place, and there are only two terms to the root.

EXERCISES

Find the square root of each of the following:

1. 294,849.	5. 3481.	9. 100,489.	13. 357.21.
2. 37,636.	6. 7569.	10. 265.69.	14. 16,641.
3. 872,356.	7. 1849.	11. 87.4225.	15. 32,761.
4. 599,076.	8. 73,441.	12. 170,569.	16. 2332.89.

174. The First Digit of the Square Root of a Decimal Number. In case a number has no integral part, the first term of its square root is found as in the following examples:

1. In .1742 the first digit in the root is .4 since the square of .4 is .16, the largest square in .17.

2. In .0542 the first digit in the root is .2 since the square of .2 is .04, the largest square in .05.

3. In .0070 the first digit in the root is .08 since the square of .08 is .0064, the largest square in .0070.

4. In .0007 the first digit in the root is .02 since the square of .02 is .0004, the largest square in .0007.

From these examples we have the following

Rule. To find the first digit in the square root of a decimal number:

(1) *Divide the number into groups of two figures each counting from the decimal point toward the right, adding a zero if necessary to complete the last group.*

(2) *Take the square root of the largest square contained in the first group which is not all zeros, and prefix to it as many zeros as there are complete groups of zeros to the right of the decimal point.*

For instance, in the above Examples 1 and 2, there are no groups of zeros to the right of the decimal point. Hence the first digit in the root in each case is tenths' digit.

In Examples 3 and 4, there is one whole group of zeros to the right of the decimal point. Hence the first digit of the root is hundredths' digit.

Similarly, when there are two complete groups of zeros to the right of the decimal point, the first digit in the root is thousandths' digit.

Illustrative Example. Find the square root of .06783.

Solution.

		Square	Square Root
			$a + b + c$
		.06 78 30	.2 + .06 + .0004
$a^2 = .2^2$		$= .04$	
$2a = 2 \times .2 =$	.4	.02 78	
$b =$	.06		
$2a + b =$	.46	.02 76	$= (2a + b)b$
$2(a + b) =$	.52	.00 02 30 00	
$c =$	.0004		
$2(a + b) + c =$	.5204	.00 02 08 16	$= [2(a + b) + c]c$
		.00 02 21 84	

Explanation. According to the rule, .2 is the first term of the root because 4 is the largest square in 6 and there is no group preceding .06. The process is the same as in the case of an integral square, but special care is now needed in handling the decimal points, which is done exactly as in operations upon decimals in the process of division in arithmetic.

For instance, in finding the third term in this example, we divide .00023 by $2(.26) = .52$ and the quotient lies between .0004 and .0005. Hence $c = .0004$. Zeros are annexed to .00023 to correspond to the number of decimal places in the product $.5204 \times .0004$.

The three terms of the root thus found are $.2 + .06 + .0004 = .2604$.

To find the next term of the root we would divide .00002184 by $2(.2604) = .5208$, finding the quotient .00004. We would then add .00004 to .5208 and multiply the sum by .00004, annexing zeros to the dividend as before.

175. **Approximate Square Roots.** Evidently the process in this example may be carried on indefinitely. .2604 is an **approximation** to the square root of .06783; in fact, the square of .2604 differs from .06783 by only .00002184. The nearest approximation using three decimal places is .260. If the fourth figure were 5, or any digit greater than 5, then .261 would be the nearest approximation using three decimal places. Hence, four places must be found in order to be sure of the nearest approximation to three places; and five places must be found in order to be sure of the nearest approximation to four places, and so on.

EXERCISES

Find the square root of each of the following, correct to two decimal places:

1. 387.	5. 51.	9. 5.	13. .02.
2. 5276.	6. 3.824.	10. 7.	14. .003.
3. 2.92.	7. 2.	11. 8.	15. .5.
4. 27.29.	8. 3.	12. 11.	16. .005.

SIMPLIFYING SQUARE ROOTS

176. Approximate Square Root of a Whole Number. When we wish to approximate the square root of a number such as 8 we make use of Principle XVIII as follows:

$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$. We then find the square root of 2 and multiply by 2.

Similarly $\sqrt{12} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$,

$\sqrt{20} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$; $\sqrt{32} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}$;

$\sqrt{a^3} = \sqrt{a^2} \cdot \sqrt{a} = a\sqrt{a}$; $\sqrt{a^5} = \sqrt{a^4} \cdot \sqrt{a} = a^2\sqrt{a}$; $\sqrt{a^2y^4b} = ay^2\sqrt{b}$.

In general *if the expression under the radical sign contains a factor which is a square this factor may be removed by writing its square root before the radical sign.*

In this manner, the square roots of a few small numbers like 2, 3, 5, etc., are made to do service in finding the roots of many large numbers.

ORAL EXERCISES

Change the following so as to leave no factor which is a square under the radical sign.

1. $\sqrt{8}$.	5. $\sqrt{20}$.	9. $\sqrt{27}$.	13. $\sqrt{54}$.
2. $\sqrt{12}$.	6. $\sqrt{24}$.	10. $\sqrt{50}$.	14. $\sqrt{a^3}$.
3. $\sqrt{40}$.	7. $\sqrt{28}$.	11. $\sqrt{72}$.	15. $\sqrt{a^5}$.
4. $\sqrt{18}$.	8. $\sqrt{32}$.	12. $\sqrt{45}$.	16. $\sqrt{a^7}$.

17. $\sqrt{a^9}$.
18. $\sqrt{a^{11}}$.
19. $\sqrt{a^{17}}$.
20. $\sqrt{a^{21}}$.
21. $\sqrt{a^2b}$.
22. $\sqrt{a^4b}$.
23. $\sqrt{a^5b}$.
24. $\sqrt{a^3b}$.
25. $\sqrt{a^5b}$.
26. $\sqrt{a^2b^3}$.
27. $\sqrt{a^2b^5}$.
28. $\sqrt{a^3b^3}$.
29. $\sqrt{a^3b^4}$.
30. $\sqrt{a^3b^6}$.
31. $\sqrt{a^2b^2x}$.
32. $\sqrt{a^2b^4x}$.
33. $\sqrt{a^3b^4x}$.
34. $\sqrt{a^4b^3x^2}$.
35. $\sqrt{a^{2n}b}$.
36. $\sqrt{a^{2n}b^{4n}x}$.
37. $\sqrt{a^3b^5x}$.
38. $\sqrt{a^{2n}b^5x^3}$.
39. $\sqrt{8\,a^2b^2}$.
40. $\sqrt{18a^3b}$.

177. **Approximate Square Root of a Fraction.** A fraction is squared by squaring its numerator and its denominator separately, since $\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$. Hence, to extract the square root of a fraction, we find the square root of its numerator and denominator separately.

E.g. $\sqrt{\frac{16}{25}} = \frac{4}{5}$, since $\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}$.

When we wish to approximate the square root of a fraction, such as $\frac{1}{2}$, we make use of Principle XVIII as follows, in order to get an integer instead of a fraction under the radical sign:

$$\sqrt{\tfrac{1}{2}} = \sqrt{\tfrac{2}{4}} = \sqrt{\tfrac{1}{4} \cdot 2} = \sqrt{\tfrac{1}{4}} \cdot \sqrt{2} = \tfrac{1}{2}\sqrt{2};$$
$$\sqrt{\tfrac{2}{5}} = \sqrt{\tfrac{10}{25}} = \sqrt{\tfrac{1}{25}} \cdot \sqrt{10} = \tfrac{1}{5}\sqrt{10}.$$

In each case after multiplying both terms of the fraction by the denominator, the fraction is resolved into two factors, one of which is a perfect square.

In this way instead of getting the square root of both numerator and denominator of such a fraction as $\frac{2}{5}$, we get the square root of 10 and divide the result by 5.

In general,

$$\sqrt{\frac{1}{a}} = \sqrt{\frac{a}{a^2}} = \sqrt{\frac{1}{a^2} \cdot a} = \sqrt{\frac{1}{a^2}} \cdot \sqrt{a} = \frac{1}{a}\sqrt{a};$$

$$\sqrt{\frac{b}{a}} = \sqrt{\frac{ab}{a^2}} = \sqrt{\frac{1}{a^2} \cdot ab} = \sqrt{\frac{1}{a^2}} \cdot \sqrt{ab} = \frac{1}{a}\sqrt{ab}.$$

178. **Rationalizing the Denominator.** Changing a radical expression so as to leave no denominator under a radical sign is called **rationalizing the denominator**; thus, $\sqrt{\frac{1}{a}} = \frac{1}{a}\sqrt{a}$.

ORAL EXERCISES

Rationalize the denominator in each of the following:

1. $\sqrt{\frac{1}{2}}$.
2. $\sqrt{\frac{1}{3}}$.
3. $\sqrt{\frac{1}{5}}$.
4. $\sqrt{\frac{1}{6}}$.
5. $\sqrt{\frac{2}{3}}$.
6. $\sqrt{\frac{2}{5}}$.
7. $\sqrt{\frac{5}{6}}$.
8. $\sqrt{\frac{1}{c}}$.
9. $\sqrt{\frac{1}{b}}$.
10. $\sqrt{\frac{a}{2}}$.
11. $\sqrt{\frac{x}{3}}$.
12. $\sqrt{\frac{a}{5}}$.
13. $\sqrt{\frac{a}{6}}$.
14. $\sqrt{\frac{2}{a}}$.
15. $\sqrt{\frac{3}{a}}$.
16. $\sqrt{\frac{5}{a}}$.
17. $\sqrt{\frac{3}{2}}$.
18. $\sqrt{\frac{3}{2a}}$.
19. $\sqrt{\frac{3a}{2}}$.
20. $\sqrt{\frac{3}{5}}$.

179. In rationalizing a denominator we should multiply the terms of the fraction by the *smallest number which will make the denominator a perfect square.*

Thus, $\sqrt{\frac{1}{a^3}} = \sqrt{\frac{a}{a^4}} = \sqrt{\frac{1}{a^4}} \cdot \sqrt{a} = \frac{1}{a^2}\sqrt{a}$.

Similarly, $\sqrt{\frac{x}{a^3b^3}} = \sqrt{\frac{abx}{a^4b^4}} = \sqrt{\frac{1}{a^4b^4}} \cdot \sqrt{abx} = \frac{1}{a^2b^2}\sqrt{abx}$.

Again, $\sqrt{\frac{1}{8}} = \sqrt{\frac{2}{16}} = \sqrt{\frac{1}{16} \cdot 2} = \sqrt{\frac{1}{16}} \cdot \sqrt{2} = \frac{1}{4}\sqrt{2}$.

If the fraction is of the form $\frac{a}{\sqrt{b}}$, we multiply both numerator and denominator by $\sqrt{b}$; thus

$$\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{\sqrt{b} \cdot \sqrt{b}} = \frac{a\sqrt{b}}{b}, \quad \text{since } \sqrt{b} \cdot \sqrt{b} = \sqrt{b^2} = b.$$

E.g. $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3}\,\sqrt{3}} = \frac{\sqrt{3}}{3}$.

ORAL EXERCISES

In the manner just indicated rationalize the denominators in the following:

1. $\sqrt{\frac{1}{a^3}}$.
2. $\sqrt{\frac{1}{a^5}}$.
3. $\sqrt{\frac{1}{a^7}}$.
4. $\sqrt{\frac{b}{a^3}}$.
5. $\frac{1}{\sqrt{2}}$.
6. $\sqrt{\frac{1}{5}}$.
7. $\sqrt{\frac{1}{7}}$.
8. $\frac{1}{\sqrt{10}}$.
9. $\sqrt{\frac{1}{ab}}$.
10. $\frac{1}{\sqrt{abx}}$.
11. $\sqrt{\frac{a}{x^4y}}$.
12. $\sqrt{\frac{5}{ab}}$.

180. **Simplified Form of Radicals.** An expression in one of the forms $\sqrt{a^2b}$, $\frac{a}{\sqrt{b}}$, $\sqrt{\frac{a}{b}}$, is said to be *simplified* when it is reduced (1) so that no radical occurs in a denominator, and (2) so that no factor which is a perfect square remains under the radical sign.

Such radical expressions may always be simplified by Principle XVIII.

WRITTEN EXERCISES

Given $\sqrt{2}=1.414$, $\sqrt{3}=1.732$, $\sqrt{5}=2.236$, compute each result in Examples 1 to 15, correct to two places of decimals, without further extraction of roots:

1. $\sqrt{80}$.
2. $\sqrt{\frac{1}{8}}$.
3. $\sqrt{\frac{1}{5}}$.
4. $\sqrt{48}$.
5. $\sqrt{75}$.
6. $\sqrt{2\cdot 3}$.
7. $\sqrt{72}$.
8. $\sqrt{98}$.
9. $\sqrt{363}$.
10. $\sqrt{125}$.
11. $\sqrt{27}+\sqrt{\frac{1}{8}}$
12. $\sqrt{45}+\sqrt{\frac{1}{5}}$.
13. $\sqrt{50}-\sqrt{\frac{1}{2}}+\sqrt{8}$.
14. $\sqrt{48}+\sqrt{75}-\sqrt{3}$.
15. $\sqrt{32}+\sqrt{72}-\sqrt{18}$.

Simplify the following:

16. $\sqrt{32\,a^2b}$.
17. $\sqrt{81\,x^3b^2}$.
18. $\sqrt{50\,a^5b^4c^2}$.
19. $\sqrt{45\,x^3y^5b^3}$.
20. $\sqrt{63\,bc^5d^4}$.
21. $\sqrt{900\,ab^4c^5}$.
22. $\sqrt{500\,x^7a^3b}$.
23. $\sqrt{3x^2+6xy+3y^2}$.
24. $\sqrt{8x^2-12y^2}$.

EQUATIONS SOLVED BY SQUARE ROOTS

181. Since $2^2 = 4$ and also $(-2)^2 = 4$, it follows that the equation $x^2 = 4$ has two roots, namely $x = 2$ and $x = -2$. These are usually written $x = \pm 2$.

This solution is obtained by taking the square root of both sides, that is, by dividing both sides by the same number.

This operation may now be added to those enumerated in Principle VI for the solution of equations.

ORAL EXERCISES

Find all roots of the following equations:

1. $x^2 = 9$.
2. $x^2 = 16$.
3. $x^2 = 25$.
4. $x^2 = 36$.
5. $x^2 = 49$.
6. $x^2 = 64$.
7. $x^2 = 81$.
8. $x^2 = 100$.
9. $x^2 = 8$.
10. $x^2 = 12$.
11. $x^2 = 9a^2$.
12. $x^2 = 8a^2$.
13. $x^2 = 16a^2$.
14. $x^2 = 49b^2$.
15. $x^2 = 25a^2$.
16. $x^2 = 81a^2b^2$.
17. $x^2 = 49a^4b^2$.
18. $x^2 = 9a^2b$.
19. $x^2 = 25(a+b)^2$.
20. $x^2 = 50(a-b)^2$.
21. $x^2 = 64am^6$.
22. $x^2 = 36r^2s^4$.
23. $x^2 = 81s^8r^6$.
24. $x^2 = 50$.
25. $x^2 = 72$.
26. $x^2 = 98$.
27. $x^2 = 32$.
28. $x^2 = 49a^5$.
29. $x^2 = 36a^3b^5$.
30. $x^2 = 200a^2$.

WRITTEN EXERCISES

1. Find approximately to two decimal places the sides of a square whose area is 120.

2. Approximate to two decimals the side of a square having an area equal to that of a rectangle whose sides are 15 and 20.

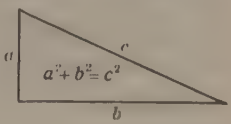
3. How many rods of fence are required to fence a square piece of land containing 50 acres, each acre containing 160 square rods?

4. A square checkerboard has an area of 324 square inches What are its dimensions?

APPLICATIONS OF SQUARE ROOT

182. The Theorem of Pythagoras. Some of the most interesting and useful applications of the square root process are concerned with the sides and areas of triangles.

The fact that the sum of the squares on the two sides of a right triangle equals the square on the hypotenuse was used in Chapter VIII. See page 151.



If a and b are the lengths of the sides, and c the length of the hypotenuse, all measured in the same unit, the theorem of Pythagoras says:

$$c^2 = a^2 + b^2. \tag{1}$$

Hence, by S,

$$a^2 = c^2 - b^2, \tag{2}$$

and

$$b^2 = c^2 - a^2. \tag{3}$$

Taking the square root of both sides in each equation, we have

$$c = \sqrt{a^2 + b^2}. \tag{4}$$

$$a = \sqrt{c^2 - b^2}. \tag{5}$$

$$b = \sqrt{c^2 - a^2}. \tag{6}$$

The negative root is omitted, since the side of a triangle cannot be a negative number. By these formulas, if any two sides of a right triangle are given, the other may be found.

E.g. if $a = 4$, $b = 3$, then, by Equation (4) above,

$$c = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$$

Again if $c = 5$, $b = 3$, then, by Equation (5) above,

$$a = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4;$$

and if $c = 5$, $a = 4$, then, by Equation (6),

$$b = \sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3.$$

Example 1. If the two sides of a right triangle are 8 and 12, find the hypotenuse correct to two decimal places.

Solution. We have $c = \sqrt{a^2 + b^2} = \sqrt{64 + 144} = \sqrt{208}$.

$$\sqrt{208} = \sqrt{16 \cdot 13} = \sqrt{16} \cdot \sqrt{13} = 4\sqrt{13} = 4(3.605) = 14.420.$$

Example 2. If the hypotenuse of a right triangle is 10 and one side is 8, find the other side.

Solution. We have $b = \sqrt{c^2 - a^2} = \sqrt{100 - 64} = \sqrt{36} = 6$.

PROBLEMS

In solving the following problems, simplify each expression under the radical sign before extracting the root. Find all results correct to two decimal places.

1. The sides about the right angle of a right triangle are each 15 inches. Find the hypotenuse.

2. The hypotenuse of a right triangle is 9 inches and one of the sides is 6 inches. Find the other side.

Hint. If x = the length of the required side, then $x^2 = a^2 - b^2 = 81 - 36$.

3. The hypotenuse of a right triangle is 25 feet and one of the other sides is 15 feet. Find the other side.

4. The hypotenuse of a right triangle is 12 inches and the other two sides are equal. Find their length.

Solution. Let s be the length of one of the equal sides.

Then,
$$s^2 + s^2 = 144.$$
$$2\,s^2 = 144.$$
$$s^2 = 72.$$
$$s = \sqrt{72} = 6\sqrt{2} = 6 \times 1.414 = 8.484.$$

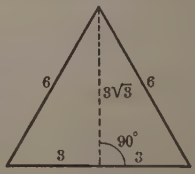
5. The hypotenuse of a right triangle is 30 feet and the other sides are equal. Find their length.

6. The hypotenuse of a right triangle is c and the sides are equal. Find their length. Solve Examples 4 and 5 by means of the formula here obtained.

7. The diagonal of a square is 8 feet. Find its area.

8. The side of an equilateral triangle is 6 inches. Find the altitude.

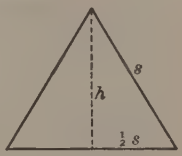
A line drawn from a vertex of an equilateral triangle perpendicular to the base meets the base at its middle point. Hence this problem becomes: The hypotenuse of a right triangle is 6 and one side is 3. Find the remaining side.



9. The side of an equilateral triangle is 10. Find the altitude.

10. The side of an equilateral triangle is s. Find the altitude.

This is equivalent to finding a side of a right triangle whose hypotenuse is s, the other side being $\frac{s}{2}$. Let h equal the altitude.



Then
$$h = \sqrt{s^2 - \left(\frac{s}{2}\right)^2} = \sqrt{s^2 - \frac{s^2}{4}}$$
$$= \sqrt{\frac{4\,s^2 - s^2}{4}} = \sqrt{\frac{3\,s^2}{4}} = \sqrt{\frac{s^2}{4} \cdot 3}$$
$$= \sqrt{\frac{s^2}{4}} \cdot \sqrt{3} = \frac{s}{2}\sqrt{3}.$$

Hence
$$h = \frac{s}{2}\sqrt{3}.$$

This formula gives the altitude of any equilateral triangle in terms of its side; namely, *the altitude of an equilateral triangle is equal to one half of the side multiplied by* $\sqrt{3}$.

By means of this formula solve Examples 8 and 9.

11. Find the altitude of an equilateral triangle whose side is $4\frac{1}{2}$. Substitute in the formula under Example 10.

12. Find the area of an equilateral triangle whose side is 5.

Since the area of a triangle is $\frac{1}{2}$ the product of the base and altitude, we first find the altitude by means of the formula under Example 10, and then multiply by $\frac{1}{2}$ the base.

13. Find the area of the equilateral triangle whose side is s. Show the result to be $\frac{s^2}{4}\sqrt{3}$.

Solution

From Example 10, the altitude is
$$h = \frac{s}{2}\sqrt{3}.$$

The area of a triangle is equal to one half the product of the base and the altitude.

Hence
$$\text{Area} = \frac{s}{2} \cdot \frac{s}{2}\sqrt{3} = \frac{s^2}{4}\sqrt{3}.$$

That is, *the area of an equilateral triangle is equal to one fourth of the square of its side multiplied by* $\sqrt{3}$.

14. If the area of an equilateral triangle is 16 square inches, find the length of the side.

Let s equal the length of the side. Then by the formula derived in Example 13, we have $16 = \frac{s^2}{4}\sqrt{3}$.

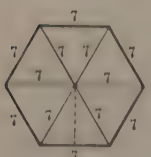
Hence, $s^2 = \frac{64}{\sqrt{3}} = \frac{64}{3}\sqrt{3} = 21.33 \times 1.732$.

15. The area of an equilateral triangle is 50 square inches. Find its side and altitude.

16. The area of an equilateral triangle is a square inches. Find the side.

Solve the equation $a = \frac{s^2}{4}\sqrt{3}$ for s, and simplify the expression, finding $s^2 = \frac{4a}{\sqrt{3}}$, and $s = \sqrt{\frac{4a\sqrt{3}}{3}} = \frac{2}{3}\sqrt{3a\sqrt{3}}$.

17. The area of an equilateral triangle is 240 square inches. Find its side. (Substitute in the formula obtained in Example 16.)



18. Find the area of a regular hexagon whose side is 7.

A regular hexagon is composed of 6 equal equilateral triangles, whose sides are each equal to the side of the hexagon (see figure). Hence this problem may be solved by finding the area of an equilateral triangle whose side is 7, and multiplying the result by 6.

19. Find the area of a regular hexagon whose side is s. Solve Example 18 by substituting in the formula obtained here.

20. The area of a regular hexagon is 108 square inches. Find its side.

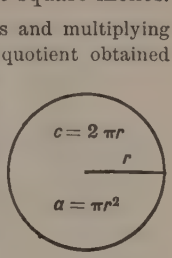
If the area of the hexagon is 108 square inches, the area of one of the equilateral triangles is 18 square inches.

21. The area of a regular hexagon is a square inches. Find its side. Solve Example 20 by substituting in the formula obtained here.

Ans. $s = \frac{1}{3}\sqrt{2a\sqrt{3}}$.

22. Find the radius of a circle whose area is 9 square inches.

The area of a circle is found by squaring the radius and multiplying by 3.1416. The number 3.1416 is approximately the quotient obtained by dividing the length of the circumference by the diameter of the circle. This quotient is represented by the Greek letter π (pronounced *pi*). In this chapter we use $3\frac{1}{7}$ as an approximation to π. This differs from the real value of π by less than .0013, and hence is accurate enough for most purposes. If a represents the area of a circle, the above rule may be written



$$a = \pi r^2.$$

Hence, if $a = 9$, $\qquad r^2 = \frac{9}{\pi} = \frac{9}{3\frac{1}{7}} = \frac{63}{22} = 2.863,$

and $\qquad r = \sqrt{2.863}.$

23. Find the radius of a circle whose area is 68 square feet.

REVIEW QUESTIONS

1. What is meant by *square root?*

2. State in words the principle

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}.$$

3. How is this principle used to simplify radicals? Show by use of this principle how to find $\sqrt{28}$, having given $\sqrt{7} = 2.645$.

4. Why is $\frac{1}{3}\sqrt{3}$ considered simpler than $\sqrt{\frac{1}{3}}$ or $\frac{1}{\sqrt{3}}$? Show how the value of the following may be approximated by finding only one square root.

$$5\sqrt{20} + 2\sqrt{45} - 3\sqrt{80} + 2\sqrt{\tfrac{1}{5}}.$$

5. Write the square of $a + b + c + d$ in such a form as to derive from it the rule for finding the square root of a polynomial.

6. In finding the square root of an arithmetic number, how is it divided into groups (1) in case of a whole number, (2) in case of a decimal?

CHAPTER XVII

FURTHER OPERATIONS ON RADICALS

183. Higher Roots. By means of an index figure the **radical sign** is made to indicate other roots than square roots.

Thus, the *cube root of* 8, or one of its three equal factors, is written $\sqrt[3]{8} = 2$. The *fourth root* of 16 is written $\sqrt[4]{16} = 2$.

Radical Expressions. Any expression which contains an indicated root is called a *radical expression.*

The expression under the radical sign is called the **radicand.**

Rational Numbers. Integers and fractions whose terms are integers are called *rational numbers.*

E.g. $2 + \sqrt[3]{5}$ is a radical expression. 5, $\frac{2}{3}$, $\frac{7.5}{100}$, are rational numbers.

A surd is an indicated root of a rational number, which is not reducible to a rational number.

E.g. $\sqrt{2}$ is a surd since it cannot be reduced to a rational number. $\sqrt[3]{4}$, $\sqrt[5]{3}$ are surds for the same reason. $\sqrt{9}$ is not a surd since $\sqrt{9} = 3$. $\sqrt{2 + \sqrt{2}}$ is not a surd since $2 + \sqrt{2}$ is not a rational number.

Order of a Surd. The *order of a surd* is indicated by the **index** of the root.

E.g. $\sqrt[3]{4}$ is a surd of the *third order*, or of index three; $\sqrt[5]{3}$ is a surd of the fifth order or of index five.

Quadratic Surd. Surd expressions containing no indicated roots except square roots are called **quadratic surds.**

E.g. $\sqrt{7}$, $\sqrt{2} + \sqrt{3}$, $3 + \sqrt{5}$, $\frac{3}{\sqrt{7} - \sqrt{5}}$, are quadratic surds.

Mixed Surd. Entire Surd. The product of a surd and a rational factor is called a *mixed surd.* A surd which has no rational factor is called an *entire surd.*

E.g. $\sqrt{2}$ is an entire surd; $3\sqrt{2}$ is a mixed surd.

REDUCTION OF SURDS

184. By **reduction of a surd** we mean the *changing of its form without changing its value.*

185. **Reduction of a Surd to its Simplest Form.** This kind of reduction was used throughout the preceding chapter and is based on Principle XVIII; namely; $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$.

E.g. $\sqrt{80} = \sqrt{16 \cdot 5} = \sqrt{16} \cdot \sqrt{5} = 4\sqrt{5}$,

and $\sqrt{\frac{1}{3}} = \sqrt{\frac{3}{9}} = \sqrt{\frac{1}{9} \cdot 3} = \sqrt{\frac{1}{9}} \cdot \sqrt{3} = \frac{1}{3}\sqrt{3}$.

186. **Reduction of a Mixed Surd to an Entire Surd.** It is sometimes desired to place the rational coefficient of a mixed surd under the radical sign. This is called *reducing a mixed surd to an entire surd.* This is also based on Principle XVIII.

Example. Reduce $3\sqrt{2}$ to an entire surd.

Since $3 = \sqrt{9}$ we have $3\sqrt{2} = \sqrt{9} \cdot \sqrt{2} = \sqrt{9 \cdot 2} = \sqrt{18}$.

In this form, Principle XVIII may be stated thus:

The product of the square roots of two numbers is equal to the square root of the product of the numbers.

ORAL EXERCISES

Reduce each of the following to entire surds.

1. $2\sqrt{2}$.
2. $2\sqrt{5}$.
3. $3\sqrt{3}$.
4. $3\sqrt{10}$.
5. $a\sqrt{b}$.
6. $a\sqrt{ab}$.
7. $x\sqrt{xy}$.
8. $ab\sqrt{ab}$.
9. $xy\sqrt{x}$.
10. $a^2\sqrt{b}$.
11. $b^2\sqrt{a^3}$.
12. $c^2\sqrt{2}$.
13. $(a+b)\sqrt{c}$.
14. $(a-b)\sqrt{c}$.
15. $(a+b)\sqrt{a-b}$.
16. $a\sqrt{a+b}$.
17. $a\sqrt{a-b}$.
18. $(a+b)\sqrt{a+b}$.
19. $(a-b)\sqrt{a-b}$
20. $(a-b)\sqrt{\frac{1}{a-b}}$
21. $3x\sqrt{\frac{1}{3x}}$.
22. $(x+y)\sqrt{\frac{1}{x+y}}$
23. $x^2\sqrt{\frac{1}{x^3}}$.
24. $ab^2\sqrt{\frac{1}{ab}}$.

187. **Simplifying a Surd of any Order.** Reductions of surds of any order may be made by means of the following:

(1) *The nth root of the product of two numbers is equal to the product of the nth roots of the numbers.*

(2) *The product of the nth roots of two numbers is equal to the nth root of the product of the numbers. In symbols:*

$$(1)\ \sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}; \qquad (2)\ \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$

Example 1. Simplify $\sqrt[3]{x^4y^5}$.

$$\sqrt[3]{x^4y^5} = \sqrt[3]{x^3y^3 \cdot xy^2} = \sqrt[3]{x^3y^3} \cdot \sqrt[3]{xy^2} = xy\sqrt[3]{xy^2}.$$

Example 2. Reduce $2\sqrt[5]{a^2}$ to an entire surd.

Since 2 may be written $\sqrt[5]{2^5}$, we have

$$2\sqrt[5]{a^2} = \sqrt[5]{2^5} \cdot \sqrt[5]{a^2} = \sqrt[5]{32\,a^2}.$$

Example 3. Simplify $\sqrt[3]{\dfrac{x^4}{3}}$

$$\sqrt[3]{\frac{x^4}{3}} = \sqrt[3]{\frac{9\,x^4}{27}} = \sqrt[3]{\frac{x^3}{27} \cdot 9\,x} = \sqrt[3]{\frac{x^3}{27}} \cdot \sqrt[3]{9\,x} = \frac{x}{3}\sqrt[3]{9\,x}.$$

188. These examples all involve the principle that *the nth root of the nth power of any number is the number itself.*

In symbols, $\sqrt[n]{a^n} = a$.

189. **Steps in Simplifying a Surd.** In simplifying a surd of any order, the first step is *to factor the radicand so that one factor shall be a perfect power of the same degree as the root indicated.*

Thus, in an indicated cube root we find the greatest factor which is a perfect cube and write its cube root before the radical sign, leaving the other factor under the radical sign.

E.g. $\sqrt[3]{16} = \sqrt[3]{8} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}.$

In an indicated fourth root we find the greatest factor which is a perfect fourth power and write its fourth root before the radical sign.

E.g. $\sqrt[4]{32} = \sqrt[4]{16} \cdot \sqrt[4]{2} = 2\sqrt[4]{2}.$

ORAL EXERCISES

Simplify the following:

1. $\sqrt[3]{16}$.
2. $\sqrt[3]{32}$.
3. $\sqrt[3]{40}$.
4. $\sqrt[3]{48}$.
5. $\sqrt[3]{x^3y^2}$.
6. $\sqrt[3]{x^6y^4}$.
7. $\sqrt[3]{ab^4}$.
8. $\sqrt[4]{32}$.
9. $\sqrt[4]{48}$.
10. $\sqrt[4]{64}$.
11. $\sqrt[5]{32}$.
12. $\sqrt[5]{64}$.
13. $\sqrt[5]{a^5b}$.
14. $\sqrt[4]{a^4b^3}$.
15. $\sqrt[4]{a^6b^5}$.
16. $\sqrt[4]{a^8b^4}$.
17. $\sqrt[6]{x^6y^6}$.
18. $\sqrt[6]{x^{12}y}$.
19. $\sqrt[6]{x^{24}y}$.
20. $\sqrt[7]{x^5y^8}$.

WRITTEN EXERCISES

Simplify the following:

1. $\sqrt[3]{a^3(a-b)^4}$.
2. $\sqrt[3]{250}$.
3. $\sqrt{147\ a^{12}b^{10}c^8}$.
4. $\sqrt{72\ x^4y^6z^8}$.
5. $\sqrt{243\ a^5x^{10}}$.
6. $\sqrt[3]{216\ x^3y^3(x+y)^3}$.
7. $\sqrt{a^3+5\ a^2}$.
8. $\sqrt[3]{125(x+y)^4}$.
9. $\sqrt{18\ a-9}$.
10. $\sqrt{98\ (x+y)^2(x-y)^2}$.
11. $\sqrt[4]{625\ x^5y\ (x-y)^4}$.
12. $\sqrt[3]{x^6-2\ x^3}$.
13. $\sqrt[3]{x^4+3\ x^3b+3\ x^2b^2+xb^3}$.
14. $\sqrt{27\ (x+y)(a-b)^3}$.
15. $\sqrt[5]{64\ b^6c^6d^{10}}$.
16. $\sqrt{(x^3+2\ x^2y+xy^2)(x^2+bx)}$.

ORAL EXERCISES

Reduce the following to entire surds:

1. $2\sqrt[3]{x}$.
2. $3\sqrt[3]{2}$.
3. $2\sqrt[4]{x}$.
4. $2\sqrt[4]{2}$.
5. $a\sqrt[3]{b}$.
6. $a\sqrt[4]{b}$.
7. $2\ a\sqrt[3]{x}$.
8. $a^2\sqrt[3]{ab}$.
9. $2\ a\sqrt[4]{x}$.
10. $2\sqrt[5]{2}$.
11. $2\sqrt[6]{2}$.
12. $ab\sqrt[3]{2}$.
13. $ab\sqrt[3]{a}$.
14. $xy^2\sqrt[3]{x}$.
15. $a^2b^2\sqrt[3]{ab}$.
16. $abx^2\sqrt[3]{2\ ab}$.
17. $2\ a^2m\sqrt[3]{3\ am^2}$.
18. $a^2m^3n\sqrt[3]{2\ a^2mn^2}$.

WRITTEN EXERCISES

Reduce the following to entire surds:

1. $2\sqrt[3]{4}$.
2. $(a+b)\sqrt{a+b}$.
3. $3\sqrt[5]{2}$.
4. $2x^3\sqrt{xy}$.
5. $2x^2\sqrt{3xyz}$.
6. $2a^2\sqrt[3]{a^2}$.
7. $a^2\sqrt[3]{b}$.
8. $2mn\sqrt{6mn}$.
9. $a^2b\sqrt[3]{a^2b}$.
10. $a\sqrt{a^2+b^2}$.
11. $\frac{2}{3}x\sqrt[3]{27x^2}$.
12. $\frac{1}{x+y}\sqrt{(x^2+y^2+2xy)xy}$.
13. $ab\sqrt{\frac{a+b}{ab}}$.
14. $ab\sqrt[5]{2ab}$.
15. $4x\sqrt[3]{2x^2y}$.
16. $-2y\sqrt[3]{6y^2}$.
17. $a^2b^3\sqrt[5]{x^3y^2}$.
18. $3xy^2\sqrt[4]{x^4y}$.
19. $-2a\sqrt[4]{a^2b}$.
20. $2xyz\sqrt[5]{xy^2z^3}$.
21. $(a+b)\sqrt{a-b}$.
22. $-abc\sqrt[5]{2abc}$.
23. $x^2y^2z^2\sqrt[5]{x^2y^2z^2}$.
24. $x\sqrt[3]{x^2+y^2}$.
25. $(x-y)\sqrt[3]{x+y}$.
26. $ab\sqrt[7]{ab}$.
27. $a(x+y)\sqrt{\frac{b}{a^2(x+y)^2}}$.
28. $\frac{xy}{x+y}\sqrt{2+\frac{x^2+y^2}{xy}}$.
29. $-abc\sqrt[9]{abc}$.

190. **Fractional Exponents.** Thus far a fraction has never been used as an exponent, and evidently it could not be so used without extending the meaning of the word *exponent*.

Thus, a^3 means $a \cdot a \cdot a$, but $a^{\frac{1}{2}}$ evidently cannot mean that a is to be multiplied by itself one half of a time.

We shall agree that the meaning attached to *fractional* exponents must be such as to make them obey the laws which govern *integral* exponents.

E.g. Just as $a^2 \cdot a^3 = a^{2+3} = a^5$, so we shall agree that $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}}$ must equal $a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$.

Since we agree that $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a$, we see that $a^{\frac{1}{2}}$ is *one of the two equal factors of* a. That is, $a^{\frac{1}{2}} = \sqrt{a}$. See § 168.

Similarly $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a$. Hence $a^{\frac{1}{3}}$ is *one of the three equal factors of* a. That is, $a^{\frac{1}{3}} = \sqrt[3]{a}$. See § 183.

Again $a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^{\frac{6}{3}} = a^2$.

Hence, $a^{\frac{2}{3}}$ is *one of the three equal factors of* a^2. That is $a^{\frac{2}{3}} = \sqrt[3]{a^2}$.

Definition of Fractional Exponent. From these examples we see that

The numerator of a fractional exponent indicates the power of the expression over which the exponent stands, and the denominator indicates what root of this power is to be taken.

Thus, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$, $a^{\frac{2}{5}} = \sqrt[5]{a^2}$, $x^{\frac{1}{2}} = \sqrt{x}$, $x^{\frac{1}{3}} = \sqrt[3]{x}$, etc.

It is also true that $a^{\frac{2}{3}} = \sqrt[3]{a^2} = (\sqrt[3]{a})^2$, that is, either the power or the root may be taken first.

E.g. $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$; and $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$.

ORAL EXERCISES

Give the equivalents of the following, using fractional exponents.

1. $\sqrt{2}$.	6. $\sqrt{x^5}$.	11. $\sqrt{a^3bc}$.	16. $\sqrt[5]{ab^2c^3}$.
2. $\sqrt{3}$.	7. $\sqrt[3]{x^2}$.	12. $\sqrt[3]{a^2b^4}$.	17. $\sqrt[5]{x^2y^3z^4}$.
3. $\sqrt[3]{2}$.	8. $\sqrt[3]{x^4}$.	13. $\sqrt[3]{m^2n^4}$.	18. $\sqrt[5]{x^3y^6z^7}$.
4. $\sqrt[3]{3}$.	9. $\sqrt[3]{x^5}$.	14. $\sqrt{nm^3y^5}$.	19. $\sqrt[7]{a^2b^4c^6}$.
5. $\sqrt{2^3}$.	10. $\sqrt[3]{x^7}$.	15. $\sqrt[4]{xy^3z^5}$.	20. $\sqrt[9]{m^2n^5p^7}$.

Operations with Fractional Exponents. If a fractional exponent is reduced to higher or lower terms, the value of the whole expression is not changed.

Thus, $4^{\frac{1}{2}} = 4^{\frac{3}{6}}$, since $4^{\frac{1}{2}} = \sqrt{4} = 2$, and $4^{\frac{3}{6}} = \sqrt[6]{4^3} = \sqrt[6]{64} = 2$.

191. Reduction of Surds to a Common Index. In reducing surds of different index to equivalent surds with a common index, fractional exponents are used as in the following examples:

1. Reduce $\sqrt[3]{5}$ and $\sqrt{3}$ to equivalent surds with the same index.

Solution. $\sqrt[3]{5} = 5^{\frac{1}{3}} = 5^{\frac{2}{6}} = \sqrt[6]{5^2} = \sqrt[6]{25}.$

$\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{3}{6}} = \sqrt[6]{3^3} = \sqrt[6]{27}.$

2. Reduce $\sqrt[5]{a}$ and $\sqrt[4]{b}$ to surds with the same index.

Solution. $\sqrt[5]{a} = a^{\frac{1}{5}} = a^{\frac{4}{20}} = \sqrt[20]{a^4}.$

$\sqrt[4]{b} = b^{\frac{1}{4}} = b^{\frac{5}{20}} = \sqrt[20]{b^5}.$

3. Reduce $\sqrt[3]{x^2}$ and $\sqrt{x^3}$ to surds with the same index.

Solution. $\sqrt[3]{x^2} = x^{\frac{2}{3}} = x^{\frac{4}{6}} = \sqrt[6]{x^4}.$

$\sqrt{x^3} = x^{\frac{3}{2}} = x^{\frac{9}{6}} = \sqrt[6]{x^9}.$

WRITTEN EXERCISES

Reduce to equivalent surds of the same order:

1. $\sqrt{a}$, $\sqrt[3]{b}$.
2. $\sqrt[3]{a}$, $\sqrt[4]{b}$.
3. $\sqrt{2}$, $\sqrt[3]{2}$.
4. $\sqrt[3]{3}$, $\sqrt{2}$.
5. $\sqrt[3]{3}$, $\sqrt[4]{3}$.
6. $\sqrt{3}$, $\sqrt[3]{3}$.
7. $\sqrt{x}$, $\sqrt[4]{x}$.
8. $\sqrt{ab}$, $\sqrt[3]{ab}$.
9. $\sqrt[3]{x^2y^2}$, $\sqrt[4]{xy}$.
10. $\sqrt{5}$, $\sqrt[3]{25}$.
11. $a^{\frac{1}{3}}$, $\sqrt{a}$, $\sqrt[4]{a}$.
12. $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$, $x^{\frac{1}{5}}$.
13. $2\sqrt{a}, 3\sqrt[3]{a}, 3\sqrt[4]{a}$.
14. $3a^{\frac{2}{3}}, 2a^{\frac{3}{4}}$.
15. $2x^{\frac{1}{3}}, 2x^{\frac{1}{4}}, 2x^{1\frac{1}{2}}$.

ADDITION AND SUBTRACTION OF SURDS

192. Similar surds are those which have the same surd factors, that is, the same index and same radicand.

E.g. $4\sqrt[3]{a^2}$ and $-2\sqrt[3]{a^2}$ are similar, while $\sqrt[3]{a}$ and $\sqrt[3]{b}$ are not similar.

Similar surds and those which can be reduced to this form may be combined into a single surd. For example:

1. $4\sqrt[3]{a^2} + 2\sqrt[3]{a^2} = 6\sqrt[3]{a^2}.$
2. $\sqrt{32} + \sqrt{72} = \sqrt{16 \cdot 2} + \sqrt{36 \cdot 2} = 4\sqrt{2} + 6\sqrt{2} = 10\sqrt{2}.$
3. $\sqrt{\tfrac{1}{3}} - \sqrt{\tfrac{3}{4}} = \sqrt{\tfrac{1}{9} \cdot 3} - \sqrt{\tfrac{1}{4} \cdot 3} = \tfrac{1}{3}\sqrt{3} - \tfrac{1}{2}\sqrt{3} = -\tfrac{1}{6}\sqrt{3}.$

ORAL EXERCISES

Combine each of the following into a single surd:

1. $3\sqrt{5}-2\sqrt{5}$.
2. $5\sqrt[3]{2}+3\sqrt[3]{2}$.
3. $2\sqrt{x}+3\sqrt{x}-\sqrt{x}$.
4. $a\sqrt{b}-c\sqrt{b}$.
5. $2\sqrt[3]{2}+7\sqrt[3]{2}-\sqrt[3]{2}$.
6. $4\sqrt[3]{a^2}+5\sqrt[3]{a^2}-3\sqrt[3]{a^2}$.
7. $12\sqrt[4]{x^3}+8\sqrt[4]{x^3}-5\sqrt[4]{x^3}$.
8. $a\sqrt[3]{b^2}+c\sqrt[3]{b^2}-d\sqrt[3]{b^2}$.

WRITTEN EXERCISES

Simplify each of the following as far as possible without approximating roots.

1. $\sqrt{27}+2\sqrt{48}-3\sqrt{75}$.
2. $\sqrt{20}+\sqrt{125}-\sqrt{180}$.
3. $3\sqrt{432}-4\sqrt{3}+\sqrt{147}$.
4. $3\sqrt{2450}-25\sqrt{2}+4\sqrt{13122}$.
5. $3y^2\sqrt{x^3z}+2\sqrt{x^5z^3}-yz^4\sqrt{\dfrac{xz}{y^2}}$.
6. $\sqrt{4x^3y}+\sqrt{25xy^3}-x\sqrt{xy}$.
7. $\sqrt{ax^2-bx^2}+\sqrt{4ar^2s^2-4br^2s^2}$.
8. $4\sqrt{\frac{3}{4}}-\frac{3}{4}\sqrt{\frac{3}{16}}-3\sqrt{27}$.
9. $2\sqrt{\frac{5}{3}}+\sqrt{60}+\sqrt{\frac{3}{5}}$.
10. $5\sqrt{3}-2\sqrt{48}+7\sqrt{108}$.
11. $\sqrt{a^3-a^2b}-\sqrt{ab^2-b^3}-\sqrt{(a+b)(a^2-b^2)}$.
12. $\sqrt{a}+3\sqrt{2a}-2\sqrt{3a}+\sqrt{4a}-\sqrt{8a}+\sqrt{12a}$.
13. $\sqrt{x^3+2x^2y+xy^2}-\sqrt{x^3-2x^2y+xy^2}-\sqrt{4xy^2}$.
14. $\sqrt{r-s}+\sqrt{16r-16s}+\sqrt{rt^2-st^2}-\sqrt{9(r-s)}$.
15. $\sqrt{(m-n)^2a}+\sqrt{(m+n)^2a}-\sqrt{am^2}+\sqrt{a(1-m)^2}-\sqrt{a}$.
16. $\sqrt{32x^2y^4}+\sqrt{162x^2y^4}-\sqrt{512x^2y^4}+\sqrt{1250x^2y^4}$.
17. $\sqrt[3]{16}+\sqrt[3]{54}-\sqrt[3]{250}+\sqrt[3]{128}$.
18. $\sqrt[3]{54}+\sqrt[3]{128}+\sqrt{128}+\sqrt{200}$.
19. $\sqrt[4]{32}+\sqrt[4]{162}+\sqrt[4]{512}-\sqrt[4]{1250}$.

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MULTIPLICATION OF SURDS

193. **Surds with the same index** are multiplied according to § 187.

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$

E.g. $\sqrt[3]{2} \cdot \sqrt[3]{4} = \sqrt[3]{2 \cdot 4} = \sqrt[3]{8} = 2.$

Surds of different index but with the same radicand are best multiplied by use of fractional exponents (§ 190).

E.g. $\sqrt[3]{2} \cdot \sqrt{2} = 2^{\frac{1}{3}} \cdot 2^{\frac{1}{2}} = 2^{\frac{1}{3}+\frac{1}{2}} = 2^{\frac{5}{6}} = \sqrt[6]{2^5}.$

Surds of different index and having different radicands must be reduced to the same order before multiplying.

E.g. $\sqrt[3]{2} \cdot \sqrt{3} = 2^{\frac{1}{3}} \cdot 3^{\frac{1}{2}} = 2^{\frac{2}{6}} \cdot 3^{\frac{3}{6}} = \sqrt[6]{2^2} \cdot \sqrt[6]{3^3} = \sqrt[6]{2^2 \cdot 3^3} = \sqrt[6]{4 \cdot 27} = \sqrt[6]{108}.$

ORAL EXERCISES

Multiply and simplify the products:

1. $\sqrt{2} \cdot \sqrt{3}.$
2. $\sqrt[5]{2} \cdot \sqrt[5]{3}.$
3. $\sqrt{7} \cdot \sqrt{3}.$
4. $\sqrt[3]{3} \cdot \sqrt[3]{9}.$
5. $\sqrt{2} \cdot \sqrt{8}.$
6. $\sqrt[5]{2} \cdot \sqrt[5]{16}.$
7. $\sqrt{5} \cdot \sqrt{7}.$
8. $\sqrt{6} \cdot \sqrt{12}.$
9. $\sqrt{18} \cdot \sqrt{8}.$
10. $\sqrt{x^3} \cdot \sqrt{x^5}.$
11. $\sqrt{a^2b} \cdot \sqrt{ab^2}.$
12. $\sqrt{rs^3t} \cdot \sqrt{r^3s^3t^3}.$
13. $\sqrt{7} \cdot \sqrt{\frac{1}{7}}.$
14. $\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{1}{3}}.$
15. $\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{6}{5}}.$
16. $2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}}.$
17. $3^{\frac{1}{2}} \cdot 3^{\frac{1}{4}}.$
18. $4^{\frac{1}{3}} \cdot 4^{\frac{1}{4}}.$

WRITTEN EXERCISES

Multiply and simplify the products.

1. $\sqrt{6} \cdot \sqrt{8}.$
2. $\sqrt{7} \cdot \sqrt{\frac{1}{14}}.$
3. $\sqrt{a^2b} \cdot \sqrt{ab^2}.$
4. $\sqrt{20} \cdot \sqrt{50}.$
5. $\sqrt{5} \cdot \sqrt{15}.$
6. $\sqrt{a} \cdot \sqrt[3]{b}.$
7. $\sqrt{6} \cdot \sqrt[3]{12}.$
8. $2\sqrt{3} \cdot 3\sqrt[3]{2}.$
9. $2\sqrt[3]{16} \cdot \sqrt{4}.$
10. $2\sqrt{12} \cdot \sqrt[3]{4}.$
11. $50^{\frac{1}{3}} \cdot 20^{\frac{1}{3}}.$
12. $2\sqrt{7} \cdot 7\sqrt{2}.$

13. $18^{\frac{1}{2}} \cdot 8^{\frac{1}{2}}$.

14. $100^{\frac{1}{3}} \cdot 40^{\frac{1}{3}}$.

15. $3^{\frac{1}{2}} \cdot 27^{\frac{1}{2}}$.

16. $\sqrt[3]{\frac{2x}{y}} \cdot \sqrt[2]{\frac{y^3}{2x^2}}$.

17. $\left(\frac{9}{x}\right)^{\frac{1}{2}}\left(\frac{4x}{9}\right)^{\frac{1}{2}}$.

18. $\left(\frac{16}{98}\right)^{\frac{1}{2}} \cdot (147)^{\frac{1}{2}}$.

19. $\sqrt{ab} \cdot \sqrt[4]{a^2b} \cdot \sqrt[8]{ab^4}$.

20. $(3abc)^{\frac{1}{2}} \cdot (bc)^{\frac{1}{3}} \cdot (a^2bc)^{\frac{1}{4}}$.

194. **Binomial Quadratic Surds** are multiplied as in the following example:

$$\begin{array}{r} 3\sqrt{2} - 4\sqrt{5} \\ 2\sqrt{2} + 3\sqrt{5} \\ \hline 6 \cdot 2 - 8\sqrt{10} \\ \quad + 9\sqrt{10} - 12 \cdot 5 \\ \hline 12 + \sqrt{10} - 60 = \sqrt{10} - 48 \end{array}$$

WRITTEN EXERCISES

1. $(\sqrt{2} - \sqrt{3})^2$.

2. $(1 + \sqrt{2})^2$.

3. $(2 - \sqrt{3})^2$.

4. $(1 + \sqrt{2})(1 - \sqrt{2})$.

5. $(\sqrt{2} + \sqrt{5})(\sqrt{2} - \sqrt{5})$.

6. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$.

7. $\sqrt{2}\,(2\sqrt{3} + 3\sqrt{8} - 5\sqrt{6})$.

8. $\sqrt{abc}\,(\sqrt{a} + \sqrt{b} + \sqrt{c})$.

9. $(3\sqrt{5} - 2\sqrt{3})(3\sqrt{5} + 2\sqrt{3})$.

10. $(1 + \sqrt{2})(1 - \sqrt{3} + \sqrt{5})$.

11. $\sqrt{3 + \sqrt{5}} \times \sqrt{3 - \sqrt{5}} = \sqrt{(3 + \sqrt{5})(3 - \sqrt{5})}$.

12. $\sqrt{4 - 2\sqrt{3}} \times \sqrt{4 + 2\sqrt{3}}$.

13. $\sqrt{8 + 3\sqrt{2}} \times \sqrt{8 - 3\sqrt{2}}$.

14. $(7\sqrt{5} - 2\sqrt{3})(7\sqrt{5} + 2\sqrt{3})$.

15. $(a + b)(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$.

16. $(3x + x\sqrt{3})(3x - x\sqrt{3})$.

17. $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{3})$.

195. **Powers and Roots of Monomial Surds** are best found by use of fractional exponents, as in the following examples:

1. $(\sqrt{5})^3 = (5^{\frac{1}{2}})^3 = 5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = 5 \cdot 5^{\frac{1}{2}} = 5\sqrt{5}$.

2. $(2^3\sqrt[4]{3})^2 = (2^3 \cdot \sqrt[4]{3})(2^3 \cdot \sqrt[4]{3}) = 2^3 \cdot 2^3 \cdot 3^{\frac{1}{4}} \cdot 3^{\frac{1}{4}}$
$= 2^6 \cdot 3^{\frac{2}{4}} = 2^6 \cdot 3^{\frac{1}{2}} = 64\sqrt{3}$.

3. $(\sqrt[5]{3}\sqrt[3]{2})^2 = (3^{\frac{1}{5}} \cdot 2^{\frac{1}{3}})(3^{\frac{1}{5}} \cdot 2^{\frac{1}{3}}) = 3^{\frac{1}{5}} \cdot 3^{\frac{1}{5}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} = 3^{\frac{2}{5}} \cdot 2^{\frac{2}{3}} = \sqrt[5]{3^2}\sqrt[3]{2^2}$.

In general, $(a^m \cdot b^n)^k = a^{km} b^{kn}$.

In the above formula k is an *integer*, but the formula is also true when k is a *fraction*.

Thus $(\sqrt{64})^{\frac{1}{3}} = \{(64)^{\frac{1}{2}}\}^{\frac{1}{3}} = 64^{\frac{1}{2} \cdot \frac{1}{3}} = 64^{\frac{1}{6}} = \sqrt[6]{64} = 2$,

which we know to be the value of $(\sqrt{64})^{\frac{1}{3}}$,

since $(\sqrt{64})^{\frac{1}{3}} = \sqrt[3]{\sqrt{64}} = \sqrt[3]{8} = 2$.

In general, $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$.

This latter formula is sometimes useful in simplifying a radical.

E.g. (1) $\sqrt[4]{125} = \sqrt{\sqrt[3]{125}} = \sqrt{5}$.

(2) $\sqrt[4]{64} = \sqrt{\sqrt{64}} = \sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$.

ORAL EXERCISES

Perform the following indicated operations:

1. $(x^{\frac{1}{2}})^2$.	10. $(x^{\frac{2}{3}})^3$.	19. $(x^{\frac{1}{2}}y^{\frac{1}{3}})^2$.
2. $(x^{\frac{1}{2}})^4$.	11. $(x^{\frac{2}{3}})^4$.	20. $(x^{\frac{1}{3}}y^{\frac{1}{2}})^6$.
3. $(a^{\frac{1}{2}}b^{\frac{1}{2}})^2$.	12. $(x^{\frac{2}{3}})^2$.	21. $(x^{\frac{2}{3}}y^{\frac{1}{3}})^3$.
4. $(a^{\frac{1}{2}}b^{\frac{1}{2}})^4$.	13. $(x^{\frac{2}{3}})^6$.	22. $(x^{\frac{2}{3}}y^{\frac{1}{3}})^6$.
5. $(a^{\frac{1}{3}})^3$.	14. $(x^{\frac{1}{4}})^2$.	23. $(a^{\frac{1}{2}}b^{\frac{1}{4}})^4$.
6. $(a^{\frac{1}{3}})^6$.	15. $(x^{\frac{1}{4}})^3$.	24. $(a^{\frac{1}{4}}b^{\frac{1}{2}})^2$.
7. $(a^{\frac{1}{3}}b^{\frac{1}{3}})^3$.	16. $(x^{\frac{1}{4}})^4$.	25. $(a^{\frac{1}{2}}b^{\frac{1}{5}})^{10}$.
8. $(a^{\frac{1}{3}}b^{\frac{1}{3}})^6$.	17. $(x^{\frac{1}{4}})^8$.	26. $(a^{\frac{1}{4}}b^{\frac{1}{4}})^6$.
9. $(4^{\frac{1}{3}})^3$.	18. $(x^{\frac{1}{4}})^6$.	27. $(a^{\frac{1}{3}}b^{\frac{1}{4}})^{12}$.

ORAL EXERCISES

1. Find the square of $\sqrt{5}$, $\sqrt[3]{2}$, $\sqrt[5]{3}$, $\sqrt[6]{x}$.
2. Find the cube of $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[5]{2}$, $\sqrt[4]{x}$.
3. Find the 4th power of $\sqrt{3}$, $\sqrt[3]{3}$, $\sqrt[4]{3}$, $\sqrt[5]{x}$.
4. Find the 5th power of $\sqrt{2}$, $\sqrt[3]{2a}$, $\sqrt{3x}$, $\sqrt[5]{4b}$.
5. Express as a single root $\sqrt{\sqrt{2}}$, $\sqrt[3]{\sqrt{5}}$, $\sqrt{\sqrt[3]{5}}$.
6. Express as a square root $\sqrt[6]{8}$, $\sqrt[4]{36}$, $\sqrt[8]{16}$.
7. Express as a single root $\sqrt[3]{\sqrt[3]{4}}$, $\sqrt[3]{\sqrt[4]{3}}$, $\sqrt[3]{\sqrt[5]{5}}$.
8. Express as a cube root $\sqrt[6]{4}$, $\sqrt[9]{8}$, $\sqrt[12]{16}$.
9. A cube root of a fifth root is what root?
10. A sixth root of a cube root is what root?

WRITTEN EXERCISES

Perform the following indicated operations:

1. $(4a^2b)^2$.
2. $(3a^3b^2c)^3$.
3. $(2\sqrt[3]{3a})^3$.
4. $(\sqrt[3]{x^2})^2$.
5. $(\sqrt[3]{a^2})^4$.
6. $(\sqrt[5]{a})^4$.
7. $(\sqrt[4]{a^3})^5$.
8. $(\sqrt[3]{a^2b^2c^2})^4$.
9. $(\sqrt[3]{bc^3d^5})^3$.
10. $(\sqrt[3]{-a^3b^5})^5$.
11. $(\sqrt{a^2b^3c^5})^3$.
12. $[(a)^{\frac{1}{2}}]^{\frac{1}{3}}$.
13. $(\sqrt{a+b})^3$.
14. $[(2x)^{\frac{1}{3}}]^4$.
15. $[(a^2)^3]^{\frac{1}{4}}$.
16. $[(ax^2)^{\frac{2}{3}}]^4$.
17. $\sqrt[3]{\sqrt[3]{x^6}}$.
18. $[(y^3)^{\frac{1}{2}}]^{\frac{1}{3}}$.
19. $[\sqrt{(x+y)^3}]^3$.
20. $[\sqrt[3]{(a-b)^2}]^4$.
21. $[x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{1}{4}}]^{12}$.
22. $[a^{\frac{2}{3}}b^{\frac{4}{5}}]^{15}$.
23. $[(\sqrt{x+y})^3]^2$.
24. $\sqrt[4]{\sqrt[3]{16}}$.
25. $\sqrt[3]{\sqrt[2]{64}}$.
26. $\sqrt{\sqrt[3]{64}}$.
27. $\sqrt{\sqrt[4]{256}}$.
28. $\sqrt[4]{\sqrt{256}}$.
29. $\sqrt{\sqrt[3]{729}}$.
30. $\sqrt[3]{\sqrt{729}}$.
31. $\sqrt[5]{\sqrt{32}}$.
32. $(\sqrt[4]{8})^2 \cdot (\sqrt[3]{4})$.
33. $\sqrt{\sqrt[3]{9}}$.
34. $[(27)^{\frac{1}{2}}]^{\frac{1}{3}}$.
35. $(\sqrt{12})^3 \cdot (\sqrt[3]{16})^2$.
36. $[\{\sqrt[3]{xy(a-b)}\}^{\frac{1}{2}}]^6$.

DIVISION OF SURDS*

196. When a divisor is a surd, it is convenient to indicate the division in the form of a fraction and then to reduce the denominator to the *rational form* as in the following example:

$$\frac{\sqrt{5}}{\sqrt{5}-\sqrt{2}}=\frac{\sqrt{5}(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}=\frac{5+\sqrt{10}}{5-2}=\frac{5+\sqrt{10}}{3}.$$

Explanation. We see that if we multiply the denominator by $\sqrt{5}+\sqrt{2}$ we shall have the product of the sum and difference of $\sqrt{5}$ and $\sqrt{2}$, which is equal to the difference of their squares.

That is, $(\sqrt{5})^2-(\sqrt{2})^2=5-2=3$, which is *rational*.

If now we multiply the numerator also by $\sqrt{5}+\sqrt{2}$, the value of the fraction will not be changed. The numerator then becomes

$$\sqrt{5}(\sqrt{5}+\sqrt{2})=(\sqrt{5})^2+\sqrt{5}\cdot\sqrt{2}=5+\sqrt{10}.$$

The success of this proceeding depends upon choosing $\sqrt{5}+\sqrt{2}$ with which to multiply the denominator so as to obtain a *rational product.*

197. Rationalizing the Denominator. This is called *rationalizing the denominator,* and the factor by which we multiply is called the *rationalizing factor.* See § 178.

If we wish to compute the approximate value of $\sqrt{5}\div(\sqrt{5}-\sqrt{2})$ above, it obviously requires less numerical work to use the form $(5+\sqrt{10})\div 3$ with rational denominator, instead of $\sqrt{5}\div(\sqrt{5}-\sqrt{2})$, which involves the extraction of two square roots and a *long* division, while the former requires the extraction of only one root and a *short* division.

To rationalize the denominator of a fraction, it is necessary first to find an expression which, multiplied by the denominator of the fraction, gives a rational product, and then to multiply both terms of the fraction by this expression.

In the case of a *monomial* or *binomial quadratic surd,* the rationalizing factor may be found at sight.

Thus, if the denominator is of the form $\sqrt{x}$ or $a\sqrt{x}$, then $\sqrt{x}$ is the rationalizing factor, since $\sqrt{x}\cdot\sqrt{x}=x$.

If the denominator is of the form $\sqrt{x}+\sqrt{y}$, then $\sqrt{x}-\sqrt{y}$ is the rationalizing factor, since $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})=x-y$.

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* Articles 196 and 197 may be omitted without destroying the continuity.

ORAL EXERCISES

Give a rationalizing factor of each of the following:

(1) $\sqrt{3x}$, (2) $\sqrt{2x^3b}$, (3) $\sqrt{8b^3c^5}$, (4) $\sqrt{a}+\sqrt{b}$,
(5) $\sqrt{a}-\sqrt{b}$, (6) $\sqrt{3a}+\sqrt{2b^3}$, (7) $\sqrt{7}-\sqrt{27}$.

ORAL EXERCISES

Rationalize the denominators of each of the following:

1. $\dfrac{1}{\sqrt{x}}$.
2. $\dfrac{1}{\sqrt{x}+1}$.
3. $\dfrac{1}{\sqrt{2}-1}$.
4. $\dfrac{1}{\sqrt{5}-\sqrt{3}}$.
5. $\dfrac{1}{\sqrt{3}+\sqrt{5}}$.
6. $\dfrac{2}{\sqrt{7}-\sqrt{3}}$.
7. $\dfrac{2}{\sqrt{a}+1}$.
8. $\dfrac{1}{\sqrt{a}+\sqrt{b}}$.
9. $\dfrac{1}{\sqrt{a}-\sqrt{b}}$.
10. $\dfrac{1}{\sqrt{xy}-\sqrt{yz}}$.
11. $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}}$.
12. $\dfrac{\sqrt{8}-\sqrt{3}}{\sqrt{3}}$.

WRITTEN EXERCISES

Rationalize the denominators of each of the following:

1. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{xy}}$.
2. $\dfrac{3\sqrt{6}+9\sqrt{2}}{2\sqrt{2}}$.
3. $\dfrac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.
4. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.
5. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.
6. $\dfrac{1}{a\sqrt{x}+b\sqrt{y}}$.
7. $\dfrac{a+b}{a\sqrt{x}-b\sqrt{y}}$.
8. $\dfrac{2+\sqrt{3}}{3-\sqrt{2}}$.
9. $\dfrac{5+\sqrt{6}}{4-\sqrt{3}}$.
10. $\dfrac{2-\sqrt{5}}{3+\sqrt{3}}$.
11. $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$.
12. $\dfrac{a-\sqrt{b}}{a+\sqrt{b}}$.
13. $\dfrac{2\sqrt{5}+1}{3\sqrt{5}-\sqrt{3}}$.
14. $\dfrac{6\sqrt{3}+4\sqrt{2}}{6\sqrt{3}-4\sqrt{2}}$.
15. $\dfrac{\sqrt{x-3}-3}{\sqrt{x-3}+3}$.
16. $\dfrac{x-\sqrt{a+b}}{x+\sqrt{a+b}}$.
17. $\dfrac{1+\sqrt{2}+\sqrt{3}}{1-\sqrt{2}+\sqrt{3}}$.
18. $\dfrac{5+\sqrt{2}-\sqrt{3}}{5+\sqrt{2}+\sqrt{3}}$.

EQUATIONS INVOLVING RADICALS

198. **Illustrative Example.** Solve the equation:

$$\sqrt{x-5}+\sqrt{x+1}=3 \tag{1}$$

By $S|\sqrt{x+1}$ $\quad \sqrt{x-5}=3-\sqrt{x+1}$ (2)

Squaring both sides $\quad x-5=9-6\sqrt{x+1}+x+1.$ (3)

Transposing, $\quad 6\sqrt{x+1}=15$

or $\quad 2\sqrt{x+1}=5$ (4)

Squaring both sides $\quad 4(x+1)=25$ (5)

Hence $\quad x=5\frac{1}{4}$ (6)

Check. Substitute $x=5\frac{1}{4}$ in (1)

If two radicals are involved, or one radical and one or more rational terms, it is best to get a radical alone on one side of the equation before squaring as in Equations (2) and (4). Note that squaring both members of an equation is equivalent to multiplying both sides by the same number.

WRITTEN EXERCISES

Solve the following equations and check each result.

1. $\sqrt{x-5}=3.$
2. $x-2=\sqrt{x^2-4}.$
3. $\sqrt{x+6}=3\sqrt{x-2}.$
4. $\sqrt{x^2-2x+8}=x-4.$
5. $\sqrt{x^2+7x-4}=\sqrt{x^2+8x-5}.$
6. $\sqrt{x^2+5}+x=5.$
7. $\sqrt{4x^2-7}+2x=7.$
8. $\sqrt{x+1}-\sqrt{x-7}=2.$
9. $\sqrt{y+4}+\sqrt{y-1}=5.$
10. $\sqrt{x}+2=\sqrt{x+16}.$
11. $5-\sqrt{x}=\sqrt{x+5}.$
12. $\sqrt{x+7}=1+\sqrt{x+2}.$
13. $\dfrac{\sqrt{x}-2}{\sqrt{x}+1}=\dfrac{\sqrt{x}+1}{\sqrt{x}+13}.$
14. $\dfrac{\sqrt{x-2}}{\sqrt{x-4}}=\dfrac{\sqrt{x+1}}{\sqrt{x-3}}.$
15. $\dfrac{1}{\sqrt{x}+1}+\dfrac{1}{\sqrt{x}-1}=\dfrac{4}{x-1}.$
16. $\dfrac{2-\sqrt{x}}{1+\sqrt{x}}=\dfrac{3+\sqrt{x}}{3-\sqrt{x}}.$

Suggestions. In Example 13 clear of fractions first; in 14 square both members and then clear of fractions; in 15 rationalize the denominators.

REVIEW QUESTIONS

1. Give examples of rational expressions, of surds, of quadratic surds.

2. What three reductions of surds are considered in this chapter?

3. How is Principle XVIII used to reduce a mixed surd to an entire surd?

4. State in words the extension of Principle XVIII which in symbols is:

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

5. How is a fractional exponent defined? Show by an example such as $16^{\frac{3}{4}} \times 16^{\frac{1}{2}}$ that the law of exponents,

$$x^a \times x^b = x^{a+b}$$

holds for fractional exponents.

6. How are fractional exponents used in reducing surds with different indices to equivalent surds with the same index?

7. What are similar surds? When can two or more surds be combined into a single surd by addition or subtraction?

8. By what principle are surds of the same index multiplied? Surds of different index but with the same radicand? Surds of different index and having different radicands?

9. State in words the principle:

$$(a^m \cdot a^n)^k = a^{km} \cdot b^{kn}.$$

10.* How do we divide by a binomial quadratic surd?

11.* Give examples to show how a fraction with a binomial surd denominator may be changed into an equivalent fraction having a rational denominator.

12. In solving an equation containing radicals, how are they removed (1) when only one radical expression is involved; (2) when two such expressions are involved?

CHAPTER XVIII

QUADRATIC EQUATIONS

199. Equations of the form $x^2 + ax + b = 0$ have already been solved in cases where the left members could be factored by inspection. See § 116. However, in a case like $x^2 + 5x + 3 = 0$, the factors of the left member cannot be found by any method thus far studied. It is therefore necessary to consider other methods for solving equations of this type.

200. Completing the Square. As a preliminary step we consider again the properties of a trinomial square. See § 97.

From $(x + a)^2 = x^2 + 2ax + a^2$ we see that the third term is the square of half the coefficient of x in the second term.

Hence, if we had given only the two terms $x^2 + 2ax$, we would have to add a^2 in order to *complete the square.*

E.g. To find the term to be added to $x^2 + 6x$ in order to *complete the square*, we take the square of half the coefficient of x, that is $3^2 = 9$. Thus, $x^2 + 6x + 9$ is *a complete square.*

In general, to complete the square in $x^2 + px$, we add $\left(\frac{1}{2}p\right)^2 = \frac{p^2}{4}$. Thus $x^2 + px + \frac{p^2}{4}$ is a complete square.

ORAL EXERCISES

Complete the trinomial square in each of the following:

1. $x^2 + 2x$.	**4.** $x^2 + 8x$.	**7.** $(3x)^2 + 2(3x)$.
2. $x^2 + 4x$.	**5.** $x^2 + 3x$.	**8.** $(2x)^2 + 4(2x)$.
3. $x^2 + 6x$.	**6.** $x^2 + 5x$.	**9.** $16x^2 + 2(4x)$.

10. How do you complete the square in $a^2 - 2ab$? Is the rule different in this case?

11. Complete the square in each of the above exercises, after replacing the sign + by −.

201. Solution of a Quadratic by Completing the Square.

Example 1. Solve the equation: $x^2 + 6x + 5 = 0.$ (1)

Transposing in (1), $x^2 + 6x = -5.$ (2)

To complete the square, we add $3^2 = 9$ to both members.

Thus we have $x^2 + 6x + 3^2 = 3^2 - 5 = 4.$ (3)

Taking square roots of both sides, $x + 3 = \pm\sqrt{4} = \pm 2.$

Hence $x = -3 + 2 = -1,$

and $x = -3 - 2 = -5.$

Example 2. Solve the equation:

$$x^2 - 12x + 42 = 56. \quad (1)$$

Transposing, $x^2 - 12x = 14.$ (2)

Completing the square by adding $(\frac{1}{2}\cdot 12)^2 = 6^2 = 36$ to both sides,

$$x^2 - 12x + 36 = 14 + 36 = 50. \quad (3)$$

Taking square roots, $x - 6 = \pm\sqrt{50} = \pm 5\sqrt{2}.$ (4)

Transposing, $x = 6 \pm 7.071.$ (5)

Hence $x = 6 + 7.071 = 13.071,$

and also $x = 6 - 7.071 = -1.071.$

The steps involved in the above solutions are:

(1) *Write the equation in the form* $x^2 + px = q.$

(2) *Complete the square by adding* $(\frac{1}{2}p)^2$ *to each member.*

(3) *Take the square root of both members of this equation.*

(4) *Solve each of the first degree equations thus obtained.*

WRITTEN EXERCISES

In solving the following quadratic equations the result may in each case be reduced so that the number remaining under the radical sign shall be 2, 3, or 5. (§ 180.) Use $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$.

1. $x^2 - 4x = 8.$
2. $x^2 = 3 - 6x.$
3. $4x = 16 - x^2.$
4. $x^2 + 6x = 9.$
5. $x^2 + 6x = 11.$
6. $x^2 - 12x = 12.$
7. $x^2 - 8x = -14.$
8. $x^2 = 2x + 1.$
9. $x^2 - 4x = 16.$
10. $x^2 = 23 + 4x.$
11. $8 = x^2 + 4x.$
12. $23 - 6x = x^2.$
13. $7 + 2x = x^2.$
14. $25 - x^2 = 5x.$
15. $x^2 + \frac{7}{3}x = 2.$

202. The Hindu Method of Completing the Square. In case the coefficient of x^2 is not unity, as in $3x^2 + 8x = 4$, both members may be divided by this coefficient, and the solution is then like that of Examples 1 and 2 on page 279.

However, the following method is sometimes desirable:

$$3x^2 + 8x = 4. \tag{1}$$

Multiplying each member of equation (1) by $4 \cdot 3 = 12$, we get

$$36x^2 + 96x = 48. \tag{2}$$

This can now be written in the form $x^2 + px = q$, namely, $(6x)^2 + 16(6x) = 48$, in which $p = 16$ and $6x$ is the unknown. Hence, we add $(\frac{1}{2}6)^2 = 8^2 = 64$ to complete the square, and get

$$(6x)^2 + 16(6x) + 64 = 48 + 64 = 112. \tag{3}$$

Taking square roots, $\quad 6x + 8 = \pm\sqrt{112} = \pm 4\sqrt{7}. \tag{4}$

Hence $\quad 6x = -8 \pm 4\sqrt{7},$

and $\quad x = -\frac{4}{3} \pm \frac{2}{3}\sqrt{7}. \tag{5}$

In this solution both sides were multiplied by 4 times the original coefficient of x^2, and then the number added to complete the square was found to be *the square of the original coefficient of* x.

The advantage of this form of solution is that fractions are avoided until the last step, and *the number added to complete the square is equal to the square of the coefficient of* x *in the original equation.*

This is called the **Hindu method** of completing the square because it was first used by the Hindus.

NOTE. — Fractions would also be avoided in the above solution if equation (1) were multiplied by 3 instead of $4 \cdot 3$. This is the case only when the coefficient of x is an *even* number.

Hence, if the given equation is in the form $ax^2 + bx + c = 0$, we may complete the square *without dividing through by* a, by the following

Rule. 1. *Write the equation in the form* $\boldsymbol{ax^2 + bx = -c}$.

2. *Multiply both sides by* $\boldsymbol{4a}$.

3. *Now write the equation in the form*

$$(2ax)^2 + 2b(2ax) = -4ac.$$

4. *Complete the square by adding* $\boldsymbol{b^2}$ *to both sides.*

5. *Solve as though* $\boldsymbol{2ax}$ *were the unknown.*

EXERCISES

In the solution of the following equations the roots which contain surds may be left in simplified radical form.

1. $2x^2+3x=2$.
2. $3x^2+5x=2$.
3. $3x=9-2x^2$.
4. $6x+1=-3x^2$.
5. $2x^2=5x+3$.
6. $4x=2x^2-1$.
7. $2x^2-3x=14$.
8. $3x^2=9+2x$.
9. $4x^2=2x+1$.
10. $6x-1=3x^2$.
11. $2x^2+4x=23$.
12. $3x^2-7=4x$.
13. $2x^2-5=3x$.
14. $4x^2=6x-1$.
15. $2x=1-5x^2$.
16. $3x-20=-2x^2$.
17. $2x+3x^2=9$.
18. $4x^2-1=3x$.
19. $4x=7-2x^2$.
20. $2x+1=5x^2$.

CHECKING RESULTS IN QUADRATIC EQUATIONS

203. **Illustrative Examples.** 1. Solving $x^2-7x+12=0$, we get $x=4$ and $x=3$.

What is the *sum* of these roots? How does this compare with the coefficient of x? What is the *product* of these roots? How does this compare with the known term of the equation?

2. Solving $x^2-6x+4=0$, we get $x=3+\sqrt{5}$ and $x=3-\sqrt{5}$.

The sum of these roots, $(3+\sqrt{5})+(3-\sqrt{5})=6$, is equal to the coefficient of x with the sign changed; and the product, $(3+\sqrt{5})(3-\sqrt{5})=9-5=4$, is equal to the known term of the equation.

EXERCISES

Solve the following equations. In each case compare the product of the roots with the known term and the sum of the roots with the coefficient of x.

1. $x^2-5x+3=0$.
2. $x^2+3x+2=0$.
3. $x^2+9x+8=0$.
4. $x^2-4x-8=0$.
5. $x^2+6x-3=0$.
6. $x^2-8x=6$.

204. Relation of Roots and Coefficients. These exercises are illustrations of a **general rule** for all quadratics written in the form $x^2 + px + q = 0$, in which the coefficient of the squared term is $+1$; namely:

The sum of the roots is equal to the coefficient of x *with its sign changed,* i. e. $-p$; *and the product of the roots is equal to the known term,* q.

This may be used to check the results obtained in solving a quadratic. Note that *before applying the test* the equation must be in the form specified. See example on next page.

205. Solution of the Quadratic by Formula.

Solve the equation

$$ax^2 + bx + c = 0. \tag{1}$$

Transposing, $ax^2 + bx = -c.$

Multiplying by $4a$, $4a^2x^2 + 4abx = -4ac.$ (2)

Equation (2) may be written in the form

$$(2ax)^2 + 2b(2ax) = -4ac.$$

Completing the square as if $2ax$ were the unknown,

$$(2ax)^2 + 2b(2ax) + b^2 = b^2 - 4ac. \tag{3}$$

Taking square roots, $2ax + b = \pm\sqrt{b^2 - 4ac}.$ (4)

Transposing, $2ax = -b \pm \sqrt{b^2 - 4ac}.$

Dividing by $2a$, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$ (5)

Calling the two values of x in the result x_1 and x_2 we have,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Any quadratic equation may be reduced to the form of (1) by simplifying and collecting the coefficients of x^2 and x. Hence any quadratic equation may be solved by substituting in the formulas just obtained.

Example. Solve $2x^2 - 4x + 1 = 0$.

Substituting $a = 2$, $b = -4$, $c = 1$ in the formula,

we get $$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2}.$$

From which $$x_1 = \frac{2 + \sqrt{2}}{2} \text{ and } x_2 = \frac{2 - \sqrt{2}}{2}.$$

Check. Writing the equation in the form $x^2 + px + q = 0$, we have $x^2 - 2x + \frac{1}{2} = 0$, in which $p = -2$ and $q = \frac{1}{2}$.

Then $$x_1 + x_2 = \frac{2 + \sqrt{2}}{2} + \frac{2 - \sqrt{2}}{2} = \frac{4}{2} = 2 = -p,$$

and $$x_1 \cdot x_2 = \frac{2 + \sqrt{2}}{2} \cdot \frac{2 - \sqrt{2}}{2} = \frac{4 - 2}{4} = \frac{1}{2} = q.$$

EXERCISES

Solve the following equations by the formula and check all results by means of § 204.

1. $4x^2 + 1 = 8x$.
2. $2x^2 - 3x = 20$.
3. $2x^2 - 3 = -5x$.
4. $3x^2 + 4x = 8$.
5. $10 - 4x = 5x^2$.
6. $1 + 4x^2 = -6x$.
7. $5 - 3x = 2x^2$.
8. $7 + 4x = 2x^2$.
9. $6x^2 + 12x = 2$.
10. $6x^2 - 12x = -2$.
11. $6x^2 + 12x = -2$.
12. $6x^2 - 12x = 2$.
13. $3x^2 + 2x = 5$.
14. $2 + 3x = 2x^2$.
15. $8x + 1 = -4x^2$.
16. $8 + 4x = 3x^2$.
17. $10 + 4x = 5x^2$.
18. $2 + 5x = 3x^2$.
19. $3x + 14 = 2x^2$.
20. $3x^2 - 2x = 5$.
21. $2x^2 + 4x = 1$.
22. $4x^2 + 3x = 1$.
23. $2x^2 - 4x = 23$.
24. $2x^2 - 3x = -1$.
25. $3x + 9 = 2x^2$.
26. $3x^2 + 4x = 7$.
27. $2x - 1 = -4x^2$.
28. $5x^2 + 16x = -2$.
29. $6x^2 + 11x = 10$.
30. $5x^2 - 11x - 12 = 0$.

IMAGINARY NUMBERS*

206. There are quadratic equations which have no roots expressible in terms of the numbers of arithmetic or algebra thus far studied.

Example. Solve $x^2 - 4x = -8$.

Completing the square, $x^2 - 4x + 4 = -8 + 4 = -4$.

Taking the root and transposing, $x = 2 \pm \sqrt{-4}$.

$\sqrt{-4}$ is thus far unknown to us as a number symbol, but we will now enlarge our number system by including in it numbers of the type $\sqrt{-4}$.

207. **Definition of Imaginaries.** An even root of a negative number is called an *imaginary number*. All other numbers are called **real numbers**.

Using Principle XVIII we may reduce an imaginary number to the standard form $a\sqrt{-1}$, in which a is a real number.

E.g. $\sqrt{-4} = \sqrt{4 \times (-1)} = \sqrt{4} \cdot \sqrt{-1} = 2\sqrt{-1}$.

$\sqrt{-5} = \sqrt{5 \times (-1)} = \sqrt{5} \cdot \sqrt{-1}$.

208. **The Fundamental Operations on Imaginary Numbers.** In operating upon imaginary numbers, they should first be reduced to the standard form $a\sqrt{-1}$, and then treated as in the following examples:

(A) **Addition and Subtraction.**

(1) $\sqrt{-4} + \sqrt{-16} = \sqrt{4}\sqrt{-1} + \sqrt{16}\sqrt{-1}$

$= 2\sqrt{-1} + 4\sqrt{-1} = 6\sqrt{-1}$.

(2) $\sqrt{-9} - \sqrt{-5} = \sqrt{9}\sqrt{-1} - \sqrt{5}\sqrt{-1}$

$= 3\sqrt{-1} - \sqrt{5}\sqrt{-1} = (3 - \sqrt{5})\sqrt{-1}$.

Thus, in general, we have

$$a\sqrt{-1} + b\sqrt{-1} + c\sqrt{-1} = (a + b + c)\sqrt{-1}.$$

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* Articles 206–209 may be omitted without destroying the continuity.

(B) **Multiplication** of imaginaries is based upon the principle that the *square of the square root of a number is the number itself.*

Thus, $(\sqrt{-1})^2 = -1.$

Applying this principle again, we have

$$(\sqrt{-1})^3 = (\sqrt{-1})^2 \cdot \sqrt{-1} = -1 \cdot \sqrt{-1} = -\sqrt{-1}.$$

And still again,

$$(\sqrt{-1})^4 = (\sqrt{-1})^2 \cdot (\sqrt{-1})^2 = (-1)(-1) = +1.$$

The results

$$(\sqrt{-1})^2 = -1;\ (\sqrt{-1})^3 = -\sqrt{-1};\ (\sqrt{-1})^4 = 1,$$

should be remembered. They are used in performing operations like the following:

(1) $\sqrt{-2} \cdot \sqrt{-8} = \sqrt{2}\sqrt{-1} \cdot \sqrt{8}\sqrt{-1} = \sqrt{2}\sqrt{8}(\sqrt{-1})^2$
$\sqrt{16}(-1) = -\sqrt{16} = -4.$

(2) $\sqrt{-4} \cdot \sqrt{-9} \cdot \sqrt{-16} = \sqrt{4} \cdot \sqrt{9} \cdot \sqrt{16} \cdot (\sqrt{-1})^3$
$= 2 \cdot 3 \cdot 4(-\sqrt{-1}) = -24\sqrt{-1}.$

(3) $\sqrt{-4} \cdot \sqrt{-9} \cdot \sqrt{-16} \cdot \sqrt{-25} = 2 \cdot 3 \cdot 4 \cdot 5(\sqrt{-1})^4$
$= 120(+1) = 120.$

In general, *two imaginary factors give a negative real product, three imaginary factors give a negative imaginary product, and four imaginary factors give a positive real product.*

(C) **Division** by an imaginary number is best indicated in the form of a fraction, as in the following examples:

(1) $2 \div \sqrt{-3} = \dfrac{2}{\sqrt{-3}} = \dfrac{2\sqrt{-3}}{\sqrt{-3} \cdot \sqrt{-3}} = \dfrac{2\sqrt{-3}}{(\sqrt{-3})^2} = \dfrac{2\sqrt{-3}}{-3} = -\dfrac{2}{3}\sqrt{-3}.$

(2) $\sqrt{-9} \div \sqrt{-16} = \dfrac{\sqrt{9} \cdot \sqrt{-1}}{\sqrt{16} \cdot \sqrt{-1}} = \dfrac{3}{4} \cdot \dfrac{\sqrt{-1}}{\sqrt{-1}} = \dfrac{3}{4} \cdot 1 = \dfrac{3}{4}.$

EXERCISES

Addition and Subtraction.

1. $\sqrt{-4} + \sqrt{-9} - \sqrt{-16}.$
2. $\sqrt{-25} - \sqrt{-9} - \sqrt{-4}.$
3. $\sqrt{-2} + \sqrt{-3} - \sqrt{-1}.$
4. $\sqrt{-4} + \sqrt{-3} - \sqrt{-16}.$
5. $\sqrt{-5} + 4\sqrt{-5} + 7\sqrt{-5}.$
6. $\sqrt{-3} - 3\sqrt{-3} - 4\sqrt{-3}.$

EXERCISES

Multiplication and Division.

1. $\sqrt{-1} \cdot \sqrt{-1}$.
2. $(\sqrt{-1})^3$.
3. $(\sqrt{-1})^4$.
4. $(\sqrt{-1})^5$.
5. $\sqrt{-2} \cdot \sqrt{-2}$.
6. $\sqrt{-2} \cdot \sqrt{-3}$.
7. $\sqrt{-16} \cdot \sqrt{-9}$.
8. $\sqrt{-2} \cdot \sqrt{-18}$.
9. $\sqrt{-16} \div \sqrt{-9}$.
10. $\sqrt{-4} \div \sqrt{-5}$.
11. $\sqrt{-3} \div (\sqrt{-1})^3$.
12. $(\sqrt{-5})^2 \div (\sqrt{-2})^3$.

13. Multiply $2\sqrt{-2} - 3\sqrt{-3}$ by $3\sqrt{-2} - 2\sqrt{-3}$.

Solution.

$$\begin{array}{r} 2\sqrt{-2} - 3\sqrt{-3} \\ 3\sqrt{-2} - 2\sqrt{-3} \\ \hline 6(\sqrt{-2})^2 - 9\sqrt{2}\sqrt{3}(\sqrt{-1})^2 \\ \quad - 4\sqrt{2}\sqrt{3}(\sqrt{-1})^2 + 6(\sqrt{-3})^2 \\ \hline 6(-2) - 13\sqrt{6}(-1) + 6(-3) \end{array}$$

Simplifying, $-30 + 13\sqrt{6}$.

14. $(2 - \sqrt{-5})(2 + \sqrt{-5})$.
15. $(\sqrt{-5} + \sqrt{-3})(\sqrt{-5} - \sqrt{-3})$.
16. $(2 - 3\sqrt{-7})(2 - 5\sqrt{-7})$.
17. $(3 - \sqrt{-2})(3 - 4\sqrt{-2})$.
18. $(2 + \sqrt{-2})^2$.
19. $(3 - \sqrt{-5})^2$.
20. $(\sqrt{-3} + \sqrt{-2})^2$.
21. $(\sqrt{-2} - \sqrt{-8})^2$.

209. We may now show that the values of x found in the example of § 206 do actually satisfy the equation.

(1) Substituting $x = 2 + \sqrt{-4}$ in the equation $x^2 - 4x = -8$, we have $(2 + \sqrt{-4})^2 - 4(2 + \sqrt{-4})$, which should reduce to -8.

Squaring $2 + \sqrt{-4}$, we have $2^2 + 2 \cdot 2\sqrt{-4} + (\sqrt{-4})^2$.

Simplifying this, we get $4 + 4\sqrt{-4} - 4 = 4\sqrt{-4}$.

Finally, subtracting $4(2 + \sqrt{-4})$, we have $4\sqrt{-4} - 8 - 4\sqrt{-4} = -8$.

Hence the left side of the equation reduces to -8 when $2 + \sqrt{-4}$ is substituted for x, and the given equation is satisfied.

(2) In the same manner show that $2 - \sqrt{-4}$ satisfies the equation.

Quadratics with Imaginary Roots.

Solve the following equations. Simplify each result.

1. $7 - 3x = -5x^2$.
2. $11x - 33 = 3x^2$.
3. $14x + 8 - x^2 = 52 + 3x^2$.
4. $12 - 15x = 36 + 6x^2$.
5. $5x + x^2 + 8 = 0$.
6. $5x^2 - 10x = -6$.
7. $x^2 + 8 + 3x = -x$.
8. $11x^2 - 49x + 57 = 0$.
9. $3x^2 + 18 - 12x = 5$.
10. $37 - 4x^2 - 12x = 79$.
11. $10x^2 + 46 + 7x = 44$.
12. $45 + 3x^2 - 2x = 0$.

FRACTIONAL EQUATIONS LEADING TO QUADRATICS

Solve the following equations and check each solution by substituting in the original equation, except when the answer is given:

1. $\dfrac{3x-1}{x+1} - \dfrac{4x+3}{x-1} + \dfrac{x^2}{x^2-1} = -\dfrac{27}{x^2-1} + 1.$

2. $\dfrac{3x+5}{x-9} + \dfrac{2x+1}{x+2} = \dfrac{x-1}{x^2-7x-18}.$ *Ans.* $x = \frac{2}{5}$, or 1.

3. $\dfrac{x-4}{2x-10} - \dfrac{3x-15}{2x-6} = -\dfrac{3x^2-114}{4x^2-32x+60}.$

4. $\dfrac{6(x+4)}{x+5} - \dfrac{3(2x-1)}{x+1} = \dfrac{7}{2}.$ *Ans.* $x = 1$, or $-6\frac{1}{7}$.

5. $\dfrac{3x-4}{x-4} + \dfrac{5x-7}{2x-2} = \dfrac{9x^2-38}{2x^2-10x+8}.$ *Ans.* $x = 18\frac{1}{2}$, or 2.

6. $\dfrac{x-1}{x-2} + \dfrac{x-2}{3-x} = \dfrac{x-3}{x-4} - \dfrac{x-5}{x-6}.$ *Ans.* $x = \pm 2\sqrt{3}$.

7. $\dfrac{x+2}{x-5} + \dfrac{3x-15}{x-3} = \dfrac{3x-21}{x-3}.$

8. $\dfrac{2x-3}{-4x} + \dfrac{3x+1}{x-2} = \dfrac{4x+17}{x-2}.$ *Ans.* $x = \dfrac{-19 \pm \sqrt{345}}{4}$.

9. $\dfrac{3x-2}{2x+3} = \dfrac{2x^2+15x+28}{2x^2+5x+3} + \dfrac{2x-1}{x+1}.$

10. $\dfrac{2x-3}{2x+2} - \dfrac{x-8}{5x+2} = \dfrac{x+2}{2x+2}.$

MISCELLANEOUS QUADRATICS

Solve as many as possible of the following equations by factoring. Otherwise use the formula of § 205. In the case of surd solutions, find the results correct to two places of decimals.

1. $x^2 + 11x = 210.$
2. $5x^2 - 3x = 4.$
3. $7x + 3x^2 - 18 = 0.$
4. $2 = 5x + 7x^2.$
5. $6x - 11x^2 = -7.$
6. $-51 + 42x - 3x^2 = 0.$
7. $3x^2 + 3x = 2x + 4.$
8.* $13 - 8x + 3x^2 = 0.$
9. $2x^2 + 11x = 32x - x^2 - 27.$
10. $176 + 3x - x^2 = 2x.$
11. $x^2 + 6x - 54 = 0.$
12. $5x^2 + 9x + 12 = 4x^2 + x.$
13. $2x^2 - 4x - 25 = 0.$
14. $7x^2 + 11x = 6.$
15. $2x^2 - 11x + 5 = 0.$
16. $2x^2 - 11x = 6.$
17. $25x - 95 = x^2.$
18. $11x^2 - 42x = 2.$
19. $x^2 - 8x - 4 = x - 22.$
20.* $8x^2 + 5x = -8.$
21. $2x^2 + 3x - 3 = 12x + 2.$
22. $3x^2 - 7x = 10.$
23. $17x + 31 + 2x^2 = 0.$
24. $18 - 41x = 3 + x^2.$
25. $10x + 25 = 5 - 2x - x^2.$
26. $3x - 59 + x^2 = 0.$
27. $5x^2 + 7x - 6 = 0.$
28. $x^2 + 12 = 7x.$
29. $8x - 5x^2 = 2.$
30. $5x + 3x^2 - 22 = 0.$
31. $50 + 20x + x^2 = 5x.$
32.* $x^2 + x + 4 = 0.$
33. $20x + 2x^2 + 42 = 33x + x^2.$
34. $17x - 3x^2 = -6.$
35. $8x + 5x^2 = -2.$
36. $10 + 15x + x^2 = 26x.$
37. $3x^2 - 2x - 7 = 0.$
38. $5x^2 - 9x - 18 = 0.$
39. $7x - 7x^2 + 24 = 0.$
40.* $31 + 2x + x^2 = 0.$
41. $7x^2 + 7x - 5x^2 + 20 = x^2 - 2x + 2.$
42. $5x^2 + 3x - 7 = (x - 1)(x + 2).$
43. $(x - 3)^2 - (2x - 1)(2x + 1) + 7 = 0.$
44.* $3x + (3x - 2)^2 = 4x^2 - 1.$
45. $9x^2 - (2x - 1)^2 = (x + 3)^2.$
46. $7x^2 = 5x - (x - 2)^2 + 7.$

EQUATIONS IN THE QUADRATIC FORM

210. **Special Cases.*** Sometimes equations of a higher degree than the second may be solved by means of quadratics, as in the following examples:

Example 1. Solve $x^4 - x^2 - 12 = 0$.

Factoring, $(x^2 - 4)(x^2 + 3) = 0$.

Putting $x^2 - 4 = 0$, or $x^2 = 4$, we have $x = \pm 2$.

Putting $x^2 + 3 = 0$, or $x^2 = -3$, we have $x = \pm\sqrt{-3}$.

Checking $x = 2$, $2^4 - 2^2 - 12 = 16 - 4 - 12 = 0$.

Checking $x = \sqrt{-3}$, $(\sqrt{3})^4(\sqrt{-1})^4 - (\sqrt{3})^2(\sqrt{-1})^2 - 12$
$= 9(+1) - 3(-1) - 12 = 12 - 12 = 0$.

Let the student check for $x = -2$ and $x = -\sqrt{-3}$.

Example 2. Solve $(x^2 + 2)^2 - 7(x^2 + 2) + 12 = 0$.

Consider $x^2 + 2$ as the unknown and call it z.

Then the equation is $z^2 - 7z + 12 = 0$.

Factoring, $(z - 4)(z - 3) = 0$.

From $z - 4 = 0$ or $x^2 + 2 - 4 = 0$,

we get $x^2 = 2$ and $x = \pm\sqrt{2}$.

From $z - 3 = 0$ or $x^2 + 2 - 3 = 0$,

we get $x^2 = 1$ or $x = \pm 1$.

Check. Substitute in the original equation.

WRITTEN EXERCISES

1. $x^4 - 5x^2 + 6 = 0$.
2. $x^4 + 5x^2 - 24 = 0$.
3. $x^4 - 3x^2 - 70 = 0$.
4. $2x^4 - x^2 - 6 = 0$.
5. $3x^4 - 10x^2 + 8 = 0$.
6. $x^4 - 7x^2 + 12 = 0$.
7. $(x^2 - 5)^2 - 9(x^2 - 5) + 20 = 0$.
8. $(x^2 + x - 2)^2 + 3(x^2 + x - 2) - 10 = 0$.
9. $(x^2 - 5x + 4)^2 - (x^2 - 5x + 4) - 20 = 0$.
10. $(x^2 - 3x - 4)^2 + 7(x^2 - 3x - 4) - 8 = 0$.

* In case Articles 206–209 have been omitted, then the solutions which involve the use of imaginary numbers may be omitted here.

Equations Containing Radicals.

Example 3. Solve $x - 5 + 2\sqrt{x-5} = 8$.

Regard $\sqrt{x-5}$ as the unknown and call it z. Then $x-5$ is the square of z.

Then $z^2 = (\sqrt{x-5})^2 = x - 5.$

Hence the equation is $z^2 + 2z - 8 = 0.$

Factoring, $(z+4)(z-2) = 0.$

Hence $z = -4$ and $z = 2.$

Then $\sqrt{x-5} = -4$ and $\sqrt{x-5} = 2.$

Hence $x - 5 = 16$ and $x - 5 = 4.$

From which $x = 21$ and $x = 9.$

Note that $x = 21$ does not satisfy the original equation, but that $x = 9$ does.

Example 4. Solve $\sqrt{x} - 5\sqrt[4]{x} + 6 = 0.$

Call $\sqrt[4]{x}$ the unknown and represent it by z. Then the first term, $\sqrt{x}$, is the square of z.

Then $z^2 = (\sqrt[4]{x})^2 = (x^{\frac{1}{4}})^2 = x^{\frac{1}{2}} = \sqrt{x}.$

Hence the equation is $z^2 - 5z + 6 = 0.$

Factoring, $(z-3)(z-2) = 0.$

Hence $z = 3$ and $z = 2,$

from which $\sqrt[4]{x} = 3$ and $\sqrt[4]{x} = 2.$

Hence $x = 81$ and $x = 16.$

Check. Substitute each value of x in the given equation.

WRITTEN EXERCISES

Find all the roots and check in the first eight:

1. $x + 7\sqrt{x} - 30 = 0.$
2. $\sqrt{x} + 3\sqrt[4]{x} - 28 = 0.$
3. $x - 4\sqrt{x} - 12 = 0.$
4. $x + 5\sqrt{x} - 14 = 0.$
5. $\sqrt{x} - 3\sqrt[4]{x} - 10 = 0.$
6. $x + 3 - 4\sqrt{x+3} + 3 = 0.$
7. $x - 2 - 6\sqrt{x-2} - 7 = 0.$
8. $x + 5 - \sqrt{x+5} = 6.$
9. $x^2 + 3x + 2 + 3\sqrt{x^2+3x+2} = 0.$
10. $x^2 - 7x + 10 + \sqrt{x^2-7x+10} = 0.$
11.* $2x^2 + 3x + 6 - 2\sqrt{2x^2+3x+9} = 0.$

HISTORICAL NOTES

Quadratic Equations. The solution of the quadratic was effected by the Greek mathematician Diophantus in much the same manner that we solve it. He did not, however, admit negative roots.

Alkarismi, an Arabian, divided all quadratic equations into five classes; namely, $ax^2 = bx$, $ax^2 = c$, $ax^2 + bx = c$, $ax^2 + c = bx$, $ax^2 = bx + c$ when a, b, c are all positive numbers. This is an interesting instance of the complexities which we encounter when we fail to recognize that letters may represent negative as well as positive numbers. Alkarismi did not admit negative roots, but he recognized the possibility of a quadratic equation having two positive roots. Bhaskara, a Hindu mathematician, gives $x = 50$ and $x = -5$ as roots of $x^2 - 45x = 250$ but remarks that "The second root is not to be taken, for it is inadequate; people do not approve of negative roots." It was only after the time of Descartes (see page 240) that negative roots came into perfectly good standing.

Imaginary Roots.* The imaginary roots of equations were entirely rejected by the Ancients. Cardan in Italy (1501–1576), and Harriot in England (1560–1621), mentioned them, but simply said that they indicated that the equations which gave rise to them represented impossible conditions. It was not until the first decade of the nineteenth century when Argand in Switzerland and Gauss in Germany (see next page) gave a geometrical interpretation of imaginaries that they came to be fully understood. In this respect imaginaries shared the fate of negative numbers, which were not fully understood until Descartes in France applied them to geometry.

PROBLEMS INVOLVING QUADRATICS

Find two solutions in each case, and determine whether both are applicable to the problem.

1. Find two consecutive integers whose product is 272.

2. Find two consecutive integers whose product is 182.

3. Find two consecutive integers whose product is 992.

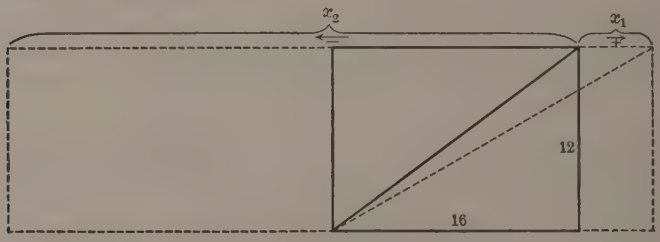
4. The sum of the squares of two consecutive integers is 145. Find the numbers.

5. Find a number such that four times the number added to 12 is equal to the square of the number.

6. Find a number such that nine times the number is equal to its square plus 18.

7. A picture is 12 inches wide and 17 inches long inside the frame. How wide is the frame if its diagonal is 25 inches?

8. A rectangle is 12 inches wide and 16 inches long. How much must be added to the length to increase the diagonal by 4 inches?



Let x = number of inches to be added to the length. The diagonal of the original rectangle is $\sqrt{12^2 + 16^2} = 20$. Hence the diagonal of the required rectangle is 24.

Then $12^2 + (16 + x)^2 = 24^2,$

or $x^2 + 32x - 176 = 0.$

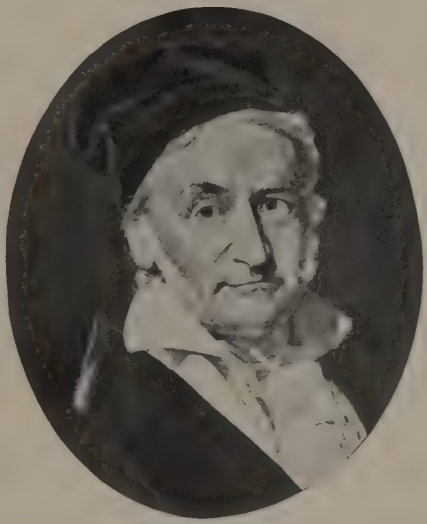
Solving, $x_1 = -16 + 12\sqrt{3} = 4.78,$

and $x_2 = -16 - 12\sqrt{3} = -36.78.$

The negative solution obtained here may be taken to mean that if the rectangle is extended in the *opposite* direction from the fixed corner, we shall get a rectangle which has the required diagonal. See the figure.

9. How much must the *width* of the rectangle in Problem 8 be extended so as to increase the diagonal by 4 inches?

10. A rectangle is 21 inches long and 20 inches wide. The length of the rectangle is decreased twice as much as the width, thereby decreasing the length of the diagonal 4 inches. Find the dimensions of the new rectangle.



Karl Friedrich Gauss was born at Brunswick, Germany, in 1777 and died at Göttingen in 1855. Justly called the greatest mathematician of modern times, he was scarcely less famous as physicist and astronomer: in fact, he was director of the observatory and professor of astronomy at Göttingen during the last forty years of his life.

He wrote on almost every phase of mathematics, and in many subjects the modern development is largely due to his genius.

Gauss was the first to explain satisfactorily the nature of imaginary quantities and to put their use in solving equations on a systematic and scientific basis. He brought into general use the symbol i for the imaginary unit $\sqrt{-1}$.

11. The distance in feet which a body falls in t seconds is expressed by $d = vt + 16\,t^2$, where v is the velocity with which the body starts.

How long will it take a body to fall from the top of the Eiffel tower, a distance of 900 feet, if it starts with a velocity of 100 feet per second?

Suggestion. Solve the equation

$$16\,t^2 + 100\,t = 900.$$

12. How long will it take a body to fall 900 feet if it starts with a velocity of 50 feet per second?

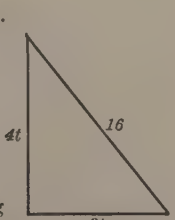
13. How long will it take a body to fall 900 feet if it starts at 10 feet per second?

14. How long will it take a bomb to fall from an aeroplane 900 feet high if it starts with no initial velocity?

15. A and B start from a certain cross roads at the same time, A going north 4 miles per hour and B going east 3 miles per hour. In how many hours will they be 16 miles apart, measuring in a straight line across country?

Solution. Let t equal the required number of hours.

Then
$$(4\,t)^2 + (3\,t)^2 = 16^2 = 256.$$
$$16\,t^2 + 9\,t^2 = 256.$$
$$25\,t^2 = 256.$$
$$5\,t = \pm 16.$$
$$t = \pm 3\tfrac{1}{5}.$$



The solution $t = -3\frac{1}{5}$ may be interpreted as meaning that if the two men had been traveling along these roads before reaching the cross road, they would have been 16 miles apart $3\frac{1}{5}$ hours before meeting.

16. In the preceding problem if A goes 5 miles per hour and B goes 4 miles per hour, in how many hours will they be 24 miles apart? *Ans.* $3\frac{3}{4}$ hours.

17. The sides of a right triangle are 6 and 8 inches respectively. How much must be added to each side in order to make the hypotenuse 20 inches? It is understood that each side is increased by the same amount.

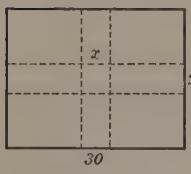
18. By what number must each side in Example 17 be *multiplied* in order to make the hypotenuse 20?

19. A park is 120 rods long and 80 rods wide. It is decided to double the area of the park, still keeping it rectangular, by adding strips of equal width to one end and one side. Find the width of the strips.

20. A fancy quilt is 72 inches long and 56 inches wide. It is decided to increase its area 10 square feet by adding a border. Find the width of the border.

21. A farmer starts cutting grain around a field 120 rods long and 70 rods wide. How wide a strip must be cut to make 12 acres?

22. In a rectangular table cover 24 by 30 inches there are two strips of drawn work of equal width running at right angles through the center of the piece. What is the width of these strips if the drawn work covers one tenth of the whole piece?

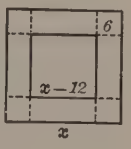


23. An athletic field is 800 feet long and 600 feet wide. The field is to be extended by the same amount in length and width so that the longest possible straight course (the diagonal) shall be 1100 feet. How much must the field be extended in each direction? *Ans.* 71.36 feet.

24. A farm is 320 rods long and 280 rods wide. There is a road running around the boundary of the farm and lying entirely within it. Find the width of the road if the area covered by it is 2384 square rods.

25. A rectangular park is 480 rods long and 360 rods wide. A walk is laid out completely around the park, and a drive is laid out through the length of the park parallel to the sides. What is the width of the walk if the drive is 3 times as wide as the walk and the combined area of the walk and the drive is 3110 square rods?

26. A square piece of tin is made into an open box, containing 864 cubic inches, by cutting out a 6-inch square from each corner of the tin and then turning up the sides. Find the dimensions of the original piece of tin.



27. A rectangular piece of tin is 8 inches longer than it is wide. By cutting out a 7-inch square from each corner and turning up the sides, an open box containing 1260 cubic inches is formed. Find the dimensions of the original piece of tin.

28. By cutting out a square 8 inches on a side from each corner of a sheet of metal and turning up the sides, we obtain an open box such that the area of the sides and ends is 4 times the area of the bottom. Find the dimensions of the original sheet if it is twice as long as it is wide.

Ans. 41.48 in. by 20.74 in.

29. An open box whose bottom is a square has a lateral area which is 400 square inches more than the area of the bottom. Find the other dimensions of the box if it is 10 inches high. (By lateral area is meant the sum of the areas of the four sides.)

30. A box whose bottom is 4 times as long as it is wide has a lateral area 600 square inches less than 4 times the area of the bottom. Find the dimensions of the bottom if the box is 6 inches high.

REVIEW QUESTIONS

1. Explain the method of solving a quadratic equation by *factoring*. Can you apply this to $(x+1)(x-2)=5$? Why?

2. Explain the method by *completing the square*. How can this be done so as to avoid fractions until the end of the solution?

3. Explain the solution of the quadratic by means of the *formula*. Why does solving the equation $\boldsymbol{ax^2+bx+c=0}$ give a formula for *all* quadratic equations? Describe this *type form* of quadratic equation in words.

4.* How many roots has a quadratic equation? Find the roots of $x^2+2x+5=0$. What are these roots called? Solve $x^2+2x-5=0$. What are these roots called?

5. How are the roots of the equation $x^2+px+q=0$ related to p and q? How may this be used in checking the solutions?

Use $2x^2-4x+1=0$ to show how the *sum* of the roots and *product* of the roots may be found from the given equation.

6. When is an equation said to be in the *quadratic form*? Which of the following are in this form?

$$x^6-3x^3-4=0;$$
$$x^2-4-2\sqrt{x^2-4}-3=0;$$
$$(x^2+1)^2+3(x^2+1)+2=0;$$
$$x^2-3x+1+\sqrt{x^2-3x+4}+5=0.$$

How can the last one be put into the quadratic form?

7. Do both roots of a quadratic equation necessarily satisfy the conditions of the problem from which such an equation may be derived? In checking the solution of a problem is it sufficient to make the test alone in the equation derived from the problem?

CHAPTER XIX

SYSTEMS OF QUADRATICS

ONE QUADRATIC AND ONE LINEAR EQUATION

211. When two simultaneous equations are given, one quadratic and one linear, they may be solved by the process of substitution, which was used (§ 150) in the case of two linear equations.

Solutions by Factoring.

Example. Solve the equations:

$$\begin{cases} x^2 - y^2 = -16. & (1) \\ x - 3y = -12. & (2) \end{cases}$$

From (2) by S, $x = 3y - 12.$ (3)

Substituting (3) in (1), $(3y - 12)^2 - y^2 = -16.$ (4)

From (4) by F, $9y^2 - 72y + 144 - y^2 = -16.$ (5)

From (5) by F, A, $8y^2 - 72y + 160 = 0.$ (6)

By D, $y^2 - 9y + 20 = 0.$ (7)

Factoring, $(y - 5)(y - 4) = 0.$ (8)

Hence, $y = 5$, and $y = 4.$ (9)

Substitute $y = 5$ in (3) and find $x = 3$.

Substitute $y = 4$ in (3) and find $x = 0$.

Therefore (1) and (3) are satisfied by the two pairs of values,

$$x = 3,\ y = 5; \qquad \text{and } x = 0,\ y = 4.$$

Check by substituting these pairs of values in (1) and (2).

Whenever the quadratic in one unknown, resulting from the substitution, can be solved by *factoring*, this method should be used as in the above solution, beginning at equation (8).

When the solution by factoring is not possible, then the formula should be used, as in the example on page 299.

EXERCISES

In the manner just illustrated solve the following:

1. $\begin{cases} x + 2y = 8, \\ 5x^2 + 12y^2 = 128. \end{cases}$

2. $\begin{cases} x + y = 1, \\ x^2 + y^2 = 1. \end{cases}$

3. $\begin{cases} 2x - y = 6, \\ 4x^2 + 5y^2 = 36. \end{cases}$

4. $\begin{cases} x + 3y = 6, \\ x^2 + 3y^2 = 12. \end{cases}$

5. $\begin{cases} x - 2y = -2, \\ x^2 - 6y^2 = 10. \end{cases}$

6. $\begin{cases} 8x - 16y = -120, \\ 7x^2 + 2y^2 = 585. \end{cases}$

7. $\begin{cases} 7x + 9y = 88, \\ 7x^2 + 9y^2 = 736. \end{cases}$

8. $\begin{cases} x - y = 6, \\ x^2 - 7y^2 = 36. \end{cases}$

9. $\begin{cases} 3x + 2y = 7, \\ 3x^2 + 8y^2 = 35. \end{cases}$

10. $\begin{cases} x - 8y = -11, \\ 3x^2 - 16y^2 = 11. \end{cases}$

11. $\begin{cases} x - y = -7, \\ 4x^2 + 3y^2 = 147. \end{cases}$

12. $\begin{cases} x - y = 2, \\ x^2 - 5y^2 = 4. \end{cases}$

13. $\begin{cases} x - y = 1, \\ 3x^2 - 2y^2 = -5. \end{cases}$

14. $\begin{cases} 5x - 7y = -28, \\ 15x^2 + 49y^2 = 784. \end{cases}$

15. $\begin{cases} 6x - 7y = 18, \\ 36x^2 - 7y^2 = 324. \end{cases}$

16. $\begin{cases} x - 9y = 2, \\ x^2 - 45y^2 = 4. \end{cases}$

17. $\begin{cases} x + y = 8, \\ 13x^2 + 3y^2 = 160. \end{cases}$

18. $\begin{cases} 2x - 5y = -16. \\ 4x^2 + 15y^2 = 256. \end{cases}$

19. $\begin{cases} 7x + 4y = 7, \\ 49x^2 - 8y^2 = 49. \end{cases}$

20. $\begin{cases} x - 3y = -12, \\ x^2 - y^2 = -16. \end{cases}$

21. $\begin{cases} x + 2y = 3, \\ 3x^2 + 4y^2 = 7. \end{cases}$

22. $\begin{cases} 3x - y = 5, \\ 3x^2 - y^2 = 11. \end{cases}$

23. $\begin{cases} x + 2y = 7, \\ 4x^2 - y^2 = 32. \end{cases}$

24. $\begin{cases} 5x + 2y = 13, \\ x + y^2 = 17. \end{cases}$

25. $\begin{cases} 4x - y = 3, \\ x^2 + 2y^2 = 54. \end{cases}$

26. $\begin{cases} x = y - 1, \\ 5x^2 + 2y = 53. \end{cases}$

27. $\begin{cases} 2y = 3x + 1, \\ 3x^2 = y^2 + 2. \end{cases}$

28. $\begin{cases} x = 3y + 3, \\ 2x^2 - 5y^2 = 67. \end{cases}$

212. Solutions by the Quadratic Formula.

Example. Solve $\begin{cases} x+y=3, & (1) \\ 3x^2-y^2=14. & (2) \end{cases}$

From (1), $y=3-x.$ (3)

Substituting in (2) and reducing,

$$2x^2+6x-23=0. \quad (4)$$

Equation (4) is in the form ax^2+bx+c, in which $a=2, b=6, c=-23$. Substituting in the formula, § 205,

$$x=\frac{-6\pm\sqrt{36-4\cdot 2(-23)}}{4}=\frac{-3\pm\sqrt{55}}{2}. \quad (5)$$

Hence $x_1=2.21$ and $x_2=-5.21$.

Substituting these values of x in (1) we have as the approximate roots,

$$\left.\begin{matrix} x_1=2.21 \\ y_1=0.79 \end{matrix}\right\} \text{ and } \left.\begin{matrix} x_2=-5.21 \\ y_2=8.21 \end{matrix}\right\}.$$

y_1 and y_2 are here used to designate those values of y which correspond to x_1 and x_2 respectively.

EXERCISES

In the above manner solve the following systems of equations, finding in each case two pairs of values.

1. $\begin{cases} x-y=1, \\ x^2+y^2=13. \end{cases}$

2. $\begin{cases} x+y=9, \\ x^2+y^2=41. \end{cases}$

3. $\begin{cases} x+y=13, \\ xy=42. \end{cases}$

4. $\begin{cases} 3x-y=5, \\ x^2+y^2=25. \end{cases}$

5. $\begin{cases} \frac{x^2}{9}-\frac{y^2}{4}=3, \\ x-y=4. \end{cases}$

6. $\begin{cases} x-y=1, \\ \frac{x^2}{36}+\frac{y^2}{16}=\frac{1}{2}. \end{cases}$

7. $\begin{cases} 2x+y=5, \\ 3x^2-5y^2=7. \end{cases}$

8. $\begin{cases} x-y=3, \\ x^2-3y^2=13. \end{cases}$

9. $\begin{cases} 3x-4y=1, \\ x^2-y^2=24. \end{cases}$

10. $\begin{cases} x+y=4, \\ 2x^2-3xy+y^2=8. \end{cases}$

11. $\begin{cases} x-y=1, \\ 4x^2+2xy-y^2=19. \end{cases}$

12. $\begin{cases} 5x+y=12, \\ 2x^2-3xy+y^2=0. \end{cases}$

13. $\begin{cases} x+y=9, \\ x^2-2y^2=-7. \end{cases}$

213. **Special Case.** Certain systems involving quadratics may be solved by special devices.

Example 1. Solve $\begin{cases} x+y=5, & (1) \\ x^2+y^2=13. & (2) \end{cases}$

Solution. Squaring (1) $x^2+2xy+y^2=25$ (3)

Subtracting (2) from (3), $\underline{x^2 \quad +y^2=13}$

$2xy \quad =12$ (4)

Subtracting (4) from (2), $\underline{x^2 \quad +y^2=13}$

$x^2-2xy+y^2=1$

Extracting the square root, $x-y=\pm 1$ (4)

Solving the system $\begin{cases} x+y=5 \\ x-y=1 \end{cases}$ we find $\begin{cases} x=3 \\ y=2. \end{cases}$

Solving the system $\begin{cases} x+y=5 \\ x-y=-1 \end{cases}$ we find $\begin{cases} x=2 \\ y=3. \end{cases}$

Example 2. Solve $\begin{cases} xy=8. & (1) \\ x^2+y^2=20. & (2) \end{cases}$

Adding twice equation (1) to (2), $x^2+2xy+y^2=36.$ (3)

Taking square roots, $x+y=\pm 6.$ (4)

Subtracting twice equation (1) from (2), $x^2-2xy+y^2=4.$ (5)

Taking square roots, $x-y=\pm 2.$ (6)

Solving the systems

$\begin{cases} x+y=6 \\ x-y=2 \end{cases}$; $\begin{cases} x+y=6 \\ x-y=-2 \end{cases}$; $\begin{cases} x+y=-6 \\ x-y=2 \end{cases}$; $\begin{cases} x+y=-6 \\ x-y=-2 \end{cases}$.

we find four pairs of solutions.

WRITTEN EXERCISES

1. Solve by the above method Examples 1, 2, 3, in the preceding exercise.

Solve by the same method:

2. $\begin{cases} xy=12, \\ x-y=1. \end{cases}$

3. $\begin{cases} xy=12, \\ x^2+y^2=25. \end{cases}$

4. $\begin{cases} xy=6, \\ x^2-xy+y^2=7. \end{cases}$

5. $\begin{cases} xy=10, \\ x^2-xy+y^2=19. \end{cases}$

6. $\begin{cases} xy=-6, \\ x^2-3xy+y^2=31. \end{cases}$

7. $\begin{cases} xy=4, \\ 2x^2+5xy+2y^2=54. \end{cases}$

A SYSTEM OF TWO QUADRATICS*

214. Homogeneous Quadratics. A quadratic equation in x and y is *homogeneous* when every term is of the second degree in x and y, that is, when every term contains x^2, xy, or y^2.

E.g. $2x^2 + 3xy + y^2 = 0$ is homogeneous in x and y.

$x^2 + 3x + y^2 = 0$ is not homogeneous, because the term $3x$ is of the first degree; and $x^2 + xy + y^2 = 5$ is not homogeneous because the term 5 contains neither x nor y.

If a quadratic equation which is homogeneous in x and y is divided through by x^2, every term will then contain $\frac{y}{x}$, and the equation may be solved for $\frac{y}{x}$ considered as the unknown, as in the following example:

Example. Solve for $\frac{y}{x}$ the equation $2x^2 + 3xy + y^2 = 0$.

Dividing by x^2, and writing $\frac{y^2}{x^2}$ first, we have

$$\frac{y^2}{x^2} + 3\frac{y}{x} + 2 = 0.$$

This is now a quadratic in which $\frac{y}{x}$ is the unknown.

Factoring, $$\left(\frac{y}{x} + 2\right)\left(\frac{y}{x} + 1\right) = 0.$$

Hence, $$\frac{y}{x} = -2 \text{ and } \frac{y}{x} = -1.$$

Hence the homogeneous quadratic equation, $2x^2 + 3xy + y^2 = 0$, is equivalent to the two linear equations, $y = -2x$ and $y = -x$.

WRITTEN EXERCISES

Solve the following equations, regarding $\frac{y}{x}$ as the unknown.

1. $y^2 - 3xy + 2x^2 = 0$.
2. $y^2 + 7xy - 30x^2 = 0$.
3. $y^2 - 11xy + 30x^2 = 0$.
4. $y^2 - 7xy - 18x^2 = 0$.
5. $3x^2 + 11xy - 20y^2 = 0$.
6. $x^2 + 8xy + 16y^2 = 0$.
7. $2x^2 + 3xy - 2y^2 = 0$.
8. $x^2 + 2xy - 63y^2 = 0$.
9. $2x^2 - 14xy - 60y^2 = 0$.
10. $24x^2 - 12xy - 12y^2 = 0$.

* Articles 214–218 may be omitted without destroying the continuity.

215. Systems in which One Equation is Homogeneous. If one of two quadratic equations in x and y is homogeneous, the complete solution may be found as in the following:

Example. Solve $\begin{cases} 2x^2+3xy+y^2=0, & (1) \\ x^2-5x+2y^2=4. & (2) \end{cases}$

Since (1) is homogeneous, we may solve it for $\frac{y}{x}$ and find, as in the example solved in § 214,

$$\frac{y}{x}=-2 \text{ and } \frac{y}{x}=-1.$$

Hence, $y=-2x$ and $y=-x$.

First. Substituting $y=-2x$ in (2), $x^2-5x+8x^2=4.$ (3)

Transposing, $9x^2-5x-4=0.$

Factoring, $(x-1)(9x+4)=0.$

Hence, $x=1$ and $x=-\frac{4}{9}$.

Hence we have

$$\begin{cases} x=1, \\ y=-2x=-2, \end{cases} \text{ and } \begin{cases} x=-\frac{4}{9}, \\ y=-2x=-2(-\frac{4}{9})=\frac{8}{9}. \end{cases}$$

Second. Substituting $y=-x$ in (2), $x^2-5x+2x^2=4.$ (4)

Transposing, $3x^2-5x-4=0.$

This is in the form $ax^2+bx+c=0$, in which $a=3$, $b=-5$, $c=-4$.

Solving by the formula, § 205, $x=\frac{5+\sqrt{73}}{6}$ and $x=\frac{5-\sqrt{73}}{6}$.

Hence we have

$$\begin{cases} x=\frac{5+\sqrt{73}}{6}, \\ y=-x=\frac{-5-\sqrt{73}}{6}. \end{cases} \qquad \begin{cases} x=\frac{5-\sqrt{73}}{6}, \\ y=-x=\frac{-5+\sqrt{73}}{6}. \end{cases}$$

216. The solutions must be carefully collected in pairs. For instance, when $y=-2x$ is substituted in equation (2) above we get equation (3), from which $x=1$ and $x=-\frac{4}{9}$. But when $y=-x$ is substituted in (2) we get (4), from which $x=\frac{5+\sqrt{73}}{6}$ and $x=\frac{5-\sqrt{73}}{6}$.

Hence $y=-2x$ goes *only* with $x=1$ and $x=-\frac{4}{9}$; while $y=-x$ goes *only* with $x=\frac{5+\sqrt{73}}{6}$ and $x=\frac{5-\sqrt{73}}{6}$.

WRITTEN EXERCISES

Solve the following systems and collect the solutions in pairs.

1. $\begin{cases} 3x^2 + xy = 85, \\ 5y^2 + 8xy - 4x^2 = 0. \end{cases}$

2. $\begin{cases} y^2 - xy + x^2 = 20, \\ 3y^2 - 13xy + 12x^2 = 0. \end{cases}$

3. $\begin{cases} y^2 + xy - x = 1, \\ 12x^2 - xy - 6y^2 = 0. \end{cases}$

4. $\begin{cases} 12x^2 + xy - y^2 = 8, \\ 40x^2 + 7xy - 3y^2 = 0. \end{cases}$

5. $\begin{cases} 9x^2 - 16y^2 = 0, \\ 2xy + 6x - 5y = 0. \end{cases}$

6. $\begin{cases} x^2 + 4xy + 5y^2 = 36, \\ x^2 + xy - 2y^2 = 0. \end{cases}$

217. **Systems in which both equations are homogeneous except for the *numerical* terms.**

Example. Solve $\begin{cases} x^2 + 3xy + 2y^2 = 15, & (1) \\ 4x^2 + 5y^2 = 24. & (2) \end{cases}$

Solution. The plan is to eliminate the *numerical* terms, as follows:

Multiplying (1) by 8 $\begin{cases} 8x^2 + 24xy + 16y^2 = 120. & (3) \end{cases}$

and (2) by 5, $\begin{cases} 20x^2 + 25y^2 = 120. & (4) \end{cases}$

Subtracting (3) from (4), $12x^2 - 24xy + 9y^2 = 0.$ (5)

Since (5) is homogeneous, $3\frac{y^2}{x^2} - 8\frac{y}{x} + 4 = 0.$ (6)

Solving (6) for $\frac{y}{x}$, $\frac{y}{x} = 2$ or $\frac{y}{x} = \frac{2}{3}$,

from which $y = 2x$ or $y = \frac{2}{3}x$.

Substituting $y = 2x$ in (2), we have $4x^2 + 20x^2 = 24$.

Hence, $x = \pm 1$.

Substituting $y = \frac{2}{3}x$ in (2), we have $4x^2 + \frac{20}{9}x^2 = 24$,

from which $x = \frac{3}{7}\sqrt{7}$ and $x = -\frac{3}{7}\sqrt{7}$.

Collecting results in pairs, we have

$$\begin{cases} x = 1 \\ y = 2 \end{cases}; \quad \begin{cases} x = -1 \\ y = -2 \end{cases}; \quad \begin{cases} x = \frac{3}{7}\sqrt{21} \\ y = \frac{2}{7}\sqrt{21} \end{cases}; \quad \begin{cases} x = -\frac{3}{7}\sqrt{21} \\ y = -\frac{2}{7}\sqrt{21} \end{cases}.$$

WRITTEN EXERCISES

Solve and arrange the results in proper pairs:

1. $\begin{cases} 2xy - x^2 = 15, \\ x^2 - xy + y^2 = 21. \end{cases}$

2. $\begin{cases} x^2 - 2xy = -12, \\ 5y^2 + 3xy = 104. \end{cases}$

3. $\begin{cases} x^2 + xy = -28, \\ y^2 + xy = 44. \end{cases}$

4. $\begin{cases} y^2 + xy = 77, \\ x^2 - xy = -12. \end{cases}$

218. Other Special Cases. A few other special cases are illustrated in the following examples, some of which include equations of higher degree than the second.

Example 1. Solve $\begin{cases} x^3 + y^3 = 9, & (1) \\ x + y = 3. & (2) \end{cases}$

Cubing (2), $x^3 + 3x^2y + 3xy^2 + y^3 = 27.$ (3)

Subtracting (1) from (3), $3x^2y + 3xy^2 = 18.$ (4)

Dividing by 3, $x^2y + xy^2 = 6.$ (5)

Multiplying (2) by xy, $x^2y + xy^2 = 3xy.$ (6)

Subtracting (5) from (6), $3xy = 6.$

$xy = 2.$ (7)

Solving (7) and (2) as in § 213, we get $\begin{cases} x = 2, \\ y = 1, \end{cases}$ and $\begin{cases} x = 1, \\ y = 2. \end{cases}$

Another method is to divide (1) by (2), obtaining

$x^2 - xy + y^2 = 3$

Squaring (2), $x^2 + 2xy + y^2 = 9$

Subtracting, $3xy = 6$

The remaining steps are the same as above.

Example 2. Solve $\begin{cases} x^4 - y^4 = 65, & (1) \\ x^2 - y^2 = 5. & (2) \end{cases}$

Dividing equation (1) by (2), $x^2 + y^2 = 13.$ (3)

Solving equations (2) and (3), we find,

$$\begin{cases} x = 3 \\ y = 2 \end{cases}; \quad \begin{cases} x = -3 \\ y = -2 \end{cases}; \quad \begin{cases} x = 3 \\ y = -2 \end{cases}; \quad \begin{cases} x = -3, \\ y = 2. \end{cases}$$

Example 3. Solve $\begin{cases} y^2 + 2xy = 7x, & (1) \\ 2y^2 - xy + x^2 = 8x. & (2) \end{cases}$

Multiplying (1) by 8, $8y^2 + 16xy = 56x.$ (3)

Multiplying (2) by 7, $14y^2 - 7xy + 7x^2 = 56x.$ (4)

Subtracting (3) from (4), $6y^2 - 23xy + 7x^2 = 0.$ (5)

Factoring, $(3y - x)(2y - 7x) = 0.$ (6)

Hence $y = \frac{x}{3}$ and $y = \frac{7x}{2}.$ (7)

Solving $y = \frac{x}{3}$ with (1), we find $\begin{cases} x = 0, \\ y = \frac{x}{3} = 0, \end{cases}$ and $\begin{cases} x = 9, \\ y = \frac{x}{3} = 3. \end{cases}$

Solving $y = \frac{7x}{2}$ with (1), we find $\begin{cases} x = 0, \\ y = \frac{7x}{2} = 0, \end{cases}$ and $\begin{cases} x = \frac{4}{11}, \\ y = \frac{7x}{2} = \frac{14}{11}. \end{cases}$

WRITTEN EXERCISES

Solve the following and arrange the results in pairs:

1. $\begin{cases} x^3+y^3=243, \\ x+y=9. \end{cases}$

2. $\begin{cases} x^4-y^4=15, \\ x^2+y^2=5. \end{cases}$

3. $\begin{cases} x^2-y^2=12, \\ x-y=2. \end{cases}$

4. $\begin{cases} x^3-y^3=26, \\ x-y=2. \end{cases}$

5. $\begin{cases} x^4-y^4=63, \\ x^2-y^2=3. \end{cases}$

6. $\begin{cases} 2x^2+4xy-5y=0, \\ 2x^2-xy+y^2-2y=0. \end{cases}$

7. $\begin{cases} x^3-y^3=54, \\ x-y=6. \end{cases}$

8. $\begin{cases} x^3+y^3=243, \\ x^2y+xy^2=162. \end{cases}$

Suggestion for Example 8. Multiply the second equation by 3 and add to the first, thus obtaining a perfect cube.

PROBLEMS INVOLVING SYSTEMS OF QUADRATICS

In each case find all the solutions and determine whether all are applicable to the problem.

1. Find two numbers whose sum is 25 and whose product is 156.

2. The sum of two numbers is 35 and their product is 300. Find the numbers.

3. The sum of two numbers is -15 and their product is -700. Find the numbers.

4. The difference of two numbers is 13 and their product is 510. Find the numbers.

5. The sum of two numbers is 18 and the sum of their reciprocals is $\frac{1}{4}$. Find the numbers.

6. The difference of two numbers is 6 and the sum of their reciprocals is $\frac{5}{36}$. Find the numbers.

7. A rectangular field is 20 rods longer than it is wide. Find its dimensions if its area is 2400 square rods.

8. A rectangular field is 20 rods longer than it is wide. Find its dimensions if its area is 50 acres.

9. The sum of the sides of a right triangle is 14 and the length of the hypotenuse is 10. Find the length of each side.

10. The length of a fence around a rectangular athletic field is 1400 feet, and the longest straight track possible on the field is 500 feet. Find the dimensions of the field.

Suggestion. Using 100 feet for the unit of measure, the equations are

$$\begin{cases} x + y = 7, \\ x^2 + y^2 = 25. \end{cases}$$

11. The difference between the sides of a right triangle is 8 and the hypotenuse is 40. Find the lengths of the sides.

12. A room is 5 feet longer than it is wide, and the distance between two opposite corners is 25 feet. Find the length and width of the room.

13. One side of a right triangle is 8 feet, and the hypotenuse is 2 feet more than twice the other side. Find the length of the hypotenuse and of the remaining side.

14. The area of a window is 2016 square inches and the perimeter of the frame is 180 inches. Find the dimensions of the window.

15. The area of a rectangular city block, including the sidewalk, is 19,200 square yards. The length of the sidewalk around the block when measured on the side next the street is 560 yards. Find the dimensions of the block.

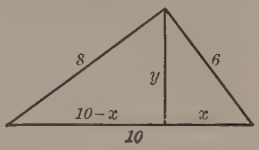
16. A farmer starts to plow around a rectangular field which contains 48 acres. The length of the first furrow around the field is 376 rods. Find the dimensions of the field.

17. A rectangular blackboard contains 38 square feet and its perimeter is 27 feet. Find the dimensions of the board.

18. The sum of the squares of two numbers plus five times their product is equal to 445; and the sum of their squares minus their product is equal to 67. Find the numbers.

19. The diagonal of a rectangular mirror inside the frame is 10 inches. The square of the diagonal of the frame is 244 inches. Find the dimensions of the mirror if the frame is 2 inches wide.

20. Find the altitude of a right triangle whose sides are 6, 8, and 10, taking the hypotenuse 10 as the base. Also find the area of this triangle.



Suggestion. Using the figure, deduce the following equations:

$$\begin{cases} x^2 + y^2 = 36, \\ (10 - x)^2 + y^2 = 64. \end{cases}$$

21. Find the altitude of a right triangle whose sides are 3, 4, and 5, taking the hypotenuse 5 as the base.

22. Find the altitude of a triangle whose sides are 10, 8, and 14, taking the side 14 as the base. Also find the area.

23. The sum of the squares of two numbers minus four times their product is -23; and the sum of their squares plus three times their product is 61. Find the numbers.

REVIEW QUESTIONS

1. Explain the method of solving a system of two equations in which one is linear and one quadratic. How many solutions are there? Explain how to solve the special case

$$\begin{cases} x + y = 5 \\ x^2 + y^2 = 13 \end{cases} \text{; also the special case } \begin{cases} xy = 16 \\ x + y = 10. \end{cases}$$

2.* When is a quadratic homogeneous in two unknowns? How can a system of two quadratics be solved when at least one of them is homogeneous? How many solutions are there?

3.* How may a system of two quadratics be solved if they are both homogeneous except for the *numerical* terms?

4.* What other special methods of solving systems of equations were used in the exercises on p. 304?

CHAPTER XX

THE BINOMIAL FORMULA*

219. We have already considered the square and the cube of a binomial, namely,

$$(a+b)^2 = a^2 + 2ab + b^2.$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

If we multiply $(a+b)^3$ by $a+b$, we have

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

In like manner,

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

The right members of these equations are called the **expansions** of the left members. By studying these expansions we see that they may be written down according to the following rule:

(1) *In the expansion of a binomial there is one more term than the number of units in the index of the power.*

(2) *The first letter of the binomial is a factor of all terms in the expansion except the* ***last.*** *Its exponent in the first term is equal to the exponent of the binomial, and it decreases by one in each succeeding term.*

(3) *The second letter is a factor of all terms in the expansion except the* ***first.*** *Its exponent in the second term is one, and in each succeeding term it increases by one.*

(4) *The coefficient of the first term is one. The coefficient of the second term is the same as the index of the power. The coefficient of each succeeding term is found by multiplying the coefficient of the preceding term by the exponent of the first letter in that term and dividing by the exponent of the second letter increased by one.*

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*Articles 219–222 may be omitted without destroying the continuity.

220. According to this rule we may write the expansions on page 308 in the following forms:

$$(a+b)^2 = a^2 + 2ab + \frac{2\cdot 1}{2} b^2.$$

$$(a+b)^3 = a^3 + 3a^2b + \frac{3\cdot 2}{2} ab^2 + \frac{3\cdot 2\cdot 1}{2\cdot 3} b^3.$$

$$(a+b)^4 = a^4 + 4a^3b + \frac{4\cdot 3}{2} a^2b^2 + \frac{4\cdot 3\cdot 2}{2\cdot 3} ab^3 + \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 3\cdot 4} b^4.$$

$$(a+b)^5 = a^5 + 5a^4b + \frac{5\cdot 4}{2} a^3b^2 + \frac{5\cdot 4\cdot 3}{2\cdot 3} a^2b^3 + \frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 3\cdot 4} ab^4 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 3\cdot 4\cdot 5} b^5.$$

ORAL EXERCISES

1. How many terms are there in the expansion of $(a+b)^5$?

2. In the expansion of $(a+b)^5$ is there any term in which a does not occur? Is there any term in which b does not occur?

3. In the expansion of $(a+b)^5$ what is the exponent of a in the first term? in the second? in the third? and so on.

4. In the expansion of $(a+b)^5$ what is the exponent of b in the second term? in the third? in the fourth? and so on.

5. In the expansion of $(a+b)^5$ state how each coefficient is obtained.

6. Ask and answer questions similar to those in Exercises 1 to 5 about the expansion of $(a+b)^6$ and $(a+b)^7$.

WRITTEN EXERCISES

Expand each of the following by the binomial formula:

1. $(x+y)^3$.	6. $(1+x)^3$.	11. $(a-b)^5$.
2. $(x+y)^4$.	7. $(1+x)^4$.	12. $(x-1)^4$.
3. $(x+y)^5$.	8. $(1+x)^5$.	13. $(x-1)^5$.
4. $(x+1)^3$.	9. $(a-b)^3$.	14. $(1-x)^4$.
5. $(x+1)^4$.	10. $(a-b)^4$.	15. $(1-x)^5$.

Suggestion for Examples 9–15: Write $a-b = a+(-b)$ and note that *even* powers of $-b$ are *positive*, while *odd* powers are *negative*.

221. Mathematical Induction. We have found by actual multiplication that the rule of § 219 holds for powers up to the fifth, and we wish to find out without continuing the multiplication whether it holds for higher powers. This is done by the process called **mathematical induction**, which is as follows:

(1) We write out the kth power according to the rule, thus:

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{2\cdot 3}a^{k-3}b^3 +, \text{ etc.}$$

(2) We find $(a+b)^{k+1}$ by multiplying the above result by $a+b$ as follows:

Multiplying by a,

$$a^{k+1} + ka^kb + \frac{k(k-1)}{2}a^{k-1}b^2 + \frac{k(k-1)(k-2)}{2\cdot 3}a^{k-2}b^3 + \cdots$$

Multiplying by b, $\quad a^kb + \quad ka^{k-1}b^2 + \quad \frac{k(k-1)}{2}a^{k-2}b^3 + \cdots$

Adding terms, $a^{k+1} + (k+1)a^kb + \left(\frac{k(k-1)}{2} + k\right)a^{k-1}b^2 +$

$$\left(\frac{k(k-1)(k-2)}{2\cdot 3} + \frac{k(k-1)}{2}\right)a^{k-2}b^3 + \cdots$$

To add $\frac{k(k-1)}{2} + k$, we use Principle I, thus

$$\frac{k(k-1)}{2} + k = k\left[\frac{k-1}{2} + 1\right] = \frac{k(k+1)}{2} = \frac{(k+1)k}{2}.$$

Similarly $\frac{k(k-1)(k-2)}{2\cdot 3} + \frac{k(k-1)}{2} = \frac{k(k-1)}{2}\left[\frac{k-2}{3} + 1\right]$

$$= \frac{k(k-1)}{2}\left[\frac{k-2+3}{3}\right] = \frac{k(k-1)(k+1)}{2\cdot 3} = \frac{(k+1)(k)(k-1)}{2\cdot 3}.$$

Hence, we have

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^kb + \frac{(k+1)(k)}{2}a^{k-1}b^2 + \frac{(k+1)(k)(k-1)}{2\cdot 3}a^{k-2}b^3 + \cdots$$

We thus see that, if the kth power of $a+b$ follows the rule, then the $(k+1)$st power also follows the rule, since the expansion of $(a+b)^{k+1}$, as just found, has $k+1$ everywhere in place of k in the expansion of $(a+b)^k$; that is, both proceed according to the same law.

Hence, if the rule for the binomial formula holds for any given exponent, it also holds for the next higher exponent.

But we have found by actual multiplication that the rule holds for the third power. Hence we know by the above argument that it holds for the fourth power, even without multiplying it out.

But the argument shows that if the rule holds for the fourth power, then it does for the fifth, and if for the fifth then for the sixth, and so on, as far as we please.

Hence we say that the **Binomial Formula** holds for any positive, integral exponent n. Stated as a formula this is:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}a^{n-4}b^4 + \cdots.$$

WRITTEN EXERCISES

1. Write out the fifth term in the expansion of $(a+b)^k$.

2. Multiply

$\frac{k(k-1)(k-2)(k-3)}{2\cdot 3\cdot 4}a^{k-4}b^4$ by a and $\frac{k(k-1)(k-2)}{2\cdot 3}a^{k-3}b^3$ by b and add the products.

222. **The General Term.** We can easily make a rule for writing the *rth term after the first.* For instance, in the 4th term after the first, the exponent of b is 4, the exponent of a is $n-4$, the last factor in the denominator is 4, and the *number* of factors in the numerator is 4.

Hence the **rth term after the first**

$$= \frac{n(n-1)(n-2)\cdots \text{to } (r \text{ factors})}{2\cdot 3\cdots r}a^{n-r}b^r.$$

E.g. the 5th term after the first in the expansion of $(a+b)^9$ is

$$\frac{9\cdot \overset{2}{\cancel{8}}\cdot 7\cdot \overset{2}{\cancel{6}}\cdot \cancel{5}}{\cancel{2}\cdot \cancel{3}\cdot \cancel{4}\cdot \cancel{5}}a^{9-5}b^5 = 126\,a^4b^5.$$

Example 1. Expand $(2x+y)^3$ by the binomial formula.

Solution. $(2x+y)^3 = (2x)^3 + 3(2x)^2y + 3(2x)y^2 + y^3$
$= 8x^3 + 12x^2y + 6y^2 + y^3.$

Example 2. Expand $\left(x^2 - \frac{1}{x}\right)^7$ by the binomial formula.

Solution. $\left(x^2-\frac{1}{x}\right)^7 = (x^2)^7 - 7(x^2)^6 \cdot \frac{1}{x} + 21(x^2)^5\left(\frac{1}{x}\right)^2 - 35(x^2)^4\left(\frac{1}{x}\right)^3$
$+ 35(x^2)^3\left(\frac{1}{x}\right)^4 - 21(x^2)^2\left(\frac{1}{x}\right)^5 + 7x^2\left(\frac{1}{x}\right)^6 - \left(\frac{1}{x}\right)^7$
$= x^{14} - 7x^{11} + 21x^8 - 35x^5 + 35x^2 - \frac{21}{x} + \frac{7}{x^4} - \frac{1}{x^7}.$

WRITTEN EXERCISES

Expand the following by the binomial formula:

1. $(x+y)^6$.
2. $(x-y)^6$.
3. $(x+y)^7$.
4. $(x-y)^7$.
5. $(2x-y)^3$.
6. $(x+2y)^3$.
7. $(x-2y)^3$.
8. $\left(a+\frac{1}{a}\right)^6$.
9. $(x-2y)^4$.
10. $(x-2y)^5$.
11. $(2x+3y)^4$.
12. $\left(\frac{a}{b}-\frac{b}{a}\right)^7$.

13. Find the 4th term after the first in the expansion of $(x+y)^8$; of $(x-y)^8$.

14. Find the 5th term after the first in $(2x+3y)^{10}$.

15. Find the 6th term in the expansion of $\left(x^2 - \frac{1}{x}\right)^9$.

16. Find the 5th term in the expansion of $\left(\frac{x^2}{y} + \frac{y}{x}\right)^7$.

17. Find the 7th term in the expansion of $(x^{\frac{1}{2}} - x^{\frac{2}{3}})^8$.

18. Find the 6th term in the expansion of $\left(x^{\frac{1}{3}} - \frac{1}{x^2}\right)^7$.

19. Find the 7th term in the expansion of $\left(a^{\frac{1}{2}} + \frac{1}{a^2}\right)^8$.

20. Find the 6th term in the expansion of $\left(a^{\frac{1}{3}} - \frac{1}{a^3}\right)^{10}$.

HISTORICAL NOTE

The Binomial Formula. Special cases of the binomial formula were known very early. The Hindus and the Arabs knew the formula for $(a+b)^2$ and $(a+b)^3$ and Vieta (1591) knew it for $(a+b)^6$. Blaise Pascal (see next page) constructed an "arithmetical triangle" from which the coefficients of any power of $a+b$ could be read when the next lower was given. The discovery of the general binomial theorem would seem to have been unavoidable from what was already known. It nevertheless remained for Sir Isaac Newton in 1676 to discover this remarkable theorem and to make use of it in many important applications.

REVIEW QUESTIONS

1. In the expansion of $(a+b)^n$ by the binomial formula (1), what are the exponents of a? of b? What are the successive coefficients?

2. What is the rth term after the first in the expansion of $(a+b)^n$?

3. How does the expansion of $(a-b)^n$ differ from that of $(a+b)^n$?

4. In the proof of the binomial formula by mathematical inductions,

(*a*) What is the first step?

(*b*) How is $(a+b)^{k+1}$ derived from $(a+b)^k$?

(*c*) How do the terms of $(a+b)^{k+1}$ compare with those of $(a+b)^k$?

(*d*) If $(a+b)^k$ follows the rule, what do we conclude in regard to $(a+b)^{k+1}$?

(*e*) Hence if the rule holds for $(a+b)^3$, what do we know of $(a+b)^4$?

(*f*) When we know that $(a+b)^4$ follows the rule, what do we conclude about $(a+b)^5$?

(*g*) How does this argument go on?

CHAPTER XXI

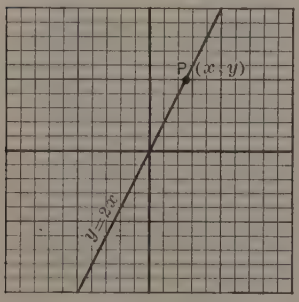
VARIABLES AND FUNCTIONS*

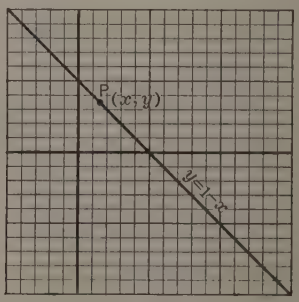
223. Definitions. In the graph of the equation $y = 2x$ we may think of the point P represented by (x, y) as moving along the line. As the point moves the values of x and y both change, subject to the fixed relation that y is always twice as great as x.

Variable. An algebraic expression whose value changes in any one problem or discussion is called a **variable.**

E.g. As P moves along the line $y=2x$, both x and y are variables.

Function. When two variables are connected by a fixed relation so that the value of one depends on the value of the other, then *one variable is said to be a function of the other.*

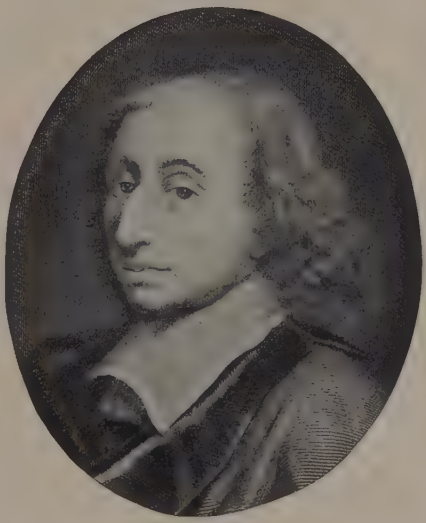




E.g. In the equation $y = 2x$, if any value is given to x, this determines the corresponding value of y. Hence y is a function of x.

Increasing Function. In the equation $y = 2x$, y is said to be an *increasing function* of x, because y *increases as* x *increases.*

* Articles 223–226 may be omitted without destroying the continuity.



Blaise Pascal was born at Clermont, France, in 1623, and died at Paris in 1662. He was a celebrated philosopher and mathematician.

At the age of twelve years, Pascal was found drawing charcoal figures on the pavement and proving theorems in geometry. At the age of sixteen he wrote a treatise on algebraic geometry which displayed great power. At nineteen he invented a machine for performing arithmetical operations, — the forerunner of our modern calculating machines.

In 1653 Pascal devised a scheme for writing down the coefficients of the binominal expansion, which he called the "Arithmetical Triangle," and which has ever since been known as the "Pascal Triangle."

Pascal wrote extensively on both philosophy and mathematics.

Decreasing Function. In the equation $y = 1 - x$, y is said to be a *decreasing function* of x, because *y decreases as x increases.*

E.g. If x increases from 0 to 1, y decreases from 1 to 0.

The graphs of $y = 2x$ and $y = 1 - x$ show the relations of the variables.

E.g. Think of the values of x as increasing from 0 toward the right. Then in $y = 2x$ the values of y *ascend*, while in $y = 1 - x$ the values of y *descend.*

ORAL EXERCISES

1. A man walks at a uniform rate of 3 miles an hour. Using d for the distance walked and t for the number of hours, express d as a function of t.

Is d an increasing or decreasing function of t?

2. A train moves at a uniform speed of 30 miles an hour. Express the distance it moves as a function of the time.

3. A pipe pours out 24 gallons of water a minute. Express the amount poured out as a function of the time.

224. Definition. If k is a constant, and if $y = kx$, then y is said to **vary as** x. This is written $y \propto x$, and is read *y varies as x.*

WRITTEN EXERCISES AND PROBLEMS

1. If $y \propto x$ and $y = 3$ when $x = 5$, find y when $x = 9$.

Solution. Let $y = kx$. If $y = 3$ when $x = 5$, then $3 = k \cdot 5$ and $k = \frac{3}{5}$. Hence, when $x = 9$, $y = kx = \frac{3}{5} \cdot 9 = \frac{27}{5} = 5\frac{2}{5}$.

2. If $y \propto x$ and $y = 7$ when $x = 3$, find y when $x = 7$.

3. If $y \propto x$ and $y = a$ when $x = b$, find y when $x = c$.

4. The circumference of a circle varies as the diameter, that is, $c = \pi \cdot d$. Find the circumference of a circle whose diameter is 10 feet if a circle whose diameter is 1 foot has a circumference of approximately 3.1416 feet.

5. The interest on a fixed sum of money varies as the length of time it is invested. If a certain sum draws \$ 145 in three months, how much will it draw in 11 months?

6. The interest on a sum of money invested for a fixed time at a fixed rate varies as the amount invested. If the interest on \$ 800 is \$ 125, what will be the interest on \$ 540 at the same rate and for the same time?

7. If a steamer uses 340 tons of coal in going 425 miles, how much will it use in going 3240 miles?

8. If an automobile uses $8\frac{1}{2}$ gallons of gasoline in going 96 miles, how many gallons will it use in going 370 miles? How far will it go in using 32 gallons?

9. The weight of an object below the earth's surface varies directly as the distance from the earth's center. If an object weighs 100 pounds at the earth's surface (4000 miles from the center), what will it weigh 2000 miles from the center? 500 miles from the center? How far from the center must it be to weigh just one pound?

225. **Inverse Variation.** If the number of men employed on a certain piece of work is doubled, the time required to finish the work is divided by 2. If the number of men is trebled, the time is divided by 3, and so on.

If n is the number of men and t the number of days required to finish a piece of work, then the relation described above may be expressed by $t = \frac{k}{n}$, where k is some fixed number depending on the piece of work to be done.

When $t = \frac{k}{n}$, we say that t varies inversely as n, and this is also written $t \propto \frac{1}{n}$.

EXERCISES

Give the results orally in Examples 1 to 4.

1. If $t = \frac{k}{n}$, and if $n = 4$ when $t = 2$, find k.

2. If $t = \frac{k}{n}$, and if $n = 7$ when $t = 3$, find k.

3. If $t = \frac{k}{n}$, and if $n = 15$ when $t = \frac{1}{5}$, find k.

4. If $t = \frac{k}{n}$, and if $n = \frac{1}{3}$ when $t = 21$, find k.

5. If a piece of work can be finished in 84 days by 16 men, how long will it take 40 men to do it?

6. If 126 men can do a piece of work in 80 days, how many men will be required to do this work in 120 days?

226. Variation as the Square of a Variable. We have seen (page 261) that $a = \pi r^2$ where r is the radius of a circle and a its area. We say that a varies directly as r^2.

It follows at once that the weight of a circular disc cut from a sheet of metal of even thickness varies as the square of the diameter.

WRITTEN EXERCISES

1. If $w = kr^2$, and if $w = 10$ when $r = 1$, find w when $r = 1\frac{1}{2}$.

2. If $y = kx^2$, and if $y = 4$ when $x = 2$, find k. Also find y when $x = 4$.

3. If $y = kx^2$, and if $y = 1$ when $x = 3$, find y when $x = 15$.

4. A circular piece of metal one foot in diameter weighs 8 pounds. How much will a circular piece weigh if it is cut from the same sheet of metal and is $2\frac{1}{2}$ feet in diameter?

5. The distance passed over by a falling body varies directly as the square of the time.

If a falling body goes 16 feet in one second, how far will it go in 7 seconds?

CHAPTER XXII

REVIEW EXERCISES

REVIEW EXERCISES FOR CHAPTER I

1. Write a formula for the area of a rectangle.

2. Write a formula for the volume of a rectangular solid.

3. If $a = 2$, $b = 3$, $c = 4$, find the value of

$$b + c - a;\ 2b - c + a;\ \frac{4b + c}{a};\ \frac{3c - 2b}{a}.$$

4. Perform the indicated operations:

$20k + 10k - 16k$; $15n + 8n - 3n$;
$50ax - 20ax - 7ax$; $63ac + 7ac - 50ac$.

5. Multiply $a + b + c$ by x; $m - n - 2$ by a; $x - y - z$ by 4.

6. Divide $42 + 12$ by 3 without first adding.

7. Divide $12x - 8y + 16z$ by 4; $6rx + 12sx$ by $6x$.

8. Multiply $3 \cdot 4 \cdot 5$ by 2 in four different ways.

9. Multiply $6ax$ by 4; $10abc$ by d; $16xyz$ by 3; $8mn$ by 7.

10. Divide $4 \cdot 8 \cdot 12$ by 2 in three different ways.

11. Divide $10abc$ by 5 c; $18mn$ by 9; $12xy$ by 3. Divide $10(2x + 4y + 6z)$ by 2 in two different ways.

12. Find the value of $3 \cdot 2 + 4 \cdot 5 - 2 \cdot 7$. What operations are performed first?

13. Find the value of $3(2 + 8 - 4) - 2(6 - 1)$.

14. If $a = 2$, $b = 1$, $c = 3$, $d = 4$, find the value of each of the following:

$c \cdot d \div a - 2b$; $6abc \div 2d$; $4ad - 3(c - 2b)$; $(10ac - 2b) \div d$.

REVIEW EXERCISES FOR CHAPTER II

1. Solve orally the equations:

$2x+3x=20$; $2(3x-x)=8$;

$16y-2(3y+y)=24$; $\frac{1}{2}n+3n-n=15$.

Solve the following equations:

2. $4n+5n+6=3n+18$.

3. $2x+4x-2x=6x-3x+15$.

4. $7(x+2)+2(4x-1)+6=6(x+4)+3(2x-1)$.

5. Six times a certain number plus 12 equals 42. What is the number?

6. For how many years must \$ 5000 be invested at 6 % to yield \$ 1800 interest?

7. The greater of two numbers is 8 times the less, and their sum is 90. Find the numbers.

8. Find a number such that when 5 times the number is subtracted from 12 times the number the remainder is 42.

9. Find three consecutive integers such that 4 times the first is 7 greater than 3 times the third.

10. A rectangle is twice as long as it is wide. The perimeter exceeds the length by 40. Find the dimensions.

11. The area of Louisiana is (nearly) 4 times that of Maryland, and the sum of their areas is 60,930 square miles. Find the (approximate) area of each state.

12. The horse power of a certain steam yacht is 12 times that of a motor boat. The sum of their horse powers is 195. Find the horse power of each.

13. At a football game there were 2000 persons. The number of women was 3 times the number of children, and the number of men was 6 times the number of children. How many men, women, and children were there?

REVIEW EXERCISES FOR CHAPTER III

Perform the following indicated operations:

1. $-8-(-9)-(+7)+8$.
2. $16+(-18)-(+2)+4$.
3. $12x+(-7x)-(-3x)+2x$.
4. $34n+(-30n)-(-7n)$.
5. $8\cdot(-3)\cdot(-1)$.
6. $[48\div(-6)]\div(-1)$.
7. $3\cdot(-2)(-3)(-1)$.
8. $(-12)\div[(-2)\div(-2)]$.
9. $(-42)\cdot(-2)\div(-7)$.
10. $[(7ab)\div(-a)]\cdot(-3)$.
11. What is meant by the average of several numbers?
12. Find the average of 20, 16, 8, 4, 0, -8, -12.
13. Find the average of -8, -32, 14, 26, -40.

Solve the following:

14. $x+6=4$.
15. $3x+12=6$.
16. $8+x=4$.
17. $3x-8=-16$.
18. $2(x+6)=3(x+5)$.
19. $2x+4=3x+8$.
20. $5x+7-2x=2x+9$.
21. $3n+2(n+4)=4n+14$.
22. $8+6x+3(2+x)=5x+26$.
23. $4(x+3)+2(3x+1)=5(x+2)+2(x+3)+19$.
24. $7(x+1)+3(2x+3)=4(x+5)+7(x+2)-8$.
25. $7(2x-3)+5(4x-1)=3(x+1)+2$.

26. If $-n$ represents a negative integer, how do you represent the next integer to the right on the number scale? the next to the left?

27. If $-2n$ represents a negative even integer, how do you represent the next even integer to the right? the next to the left?

28. Do negative numbers apply to all things to which positive numbers apply? Can there be a negative number of books on a shelf?

REVIEW EXERCISES FOR CHAPTER IV

1. Add $3x + 4y - 3z$, $5x - 2y - z$, and $3y - 5x + 7z$.
2. From $15a + 4b - 13bc$ subtract $3a - 8b + 2bc$.
3. Subtract $7x - 5y - 7a$ from $6x + 5y + 3a$.
4. From $5x - 4y - 9z$ subtract $3x - 8y + 2z$.
5. Add $5a + 3b - 2c$ and $11a - 7b + 8c$.
6. Add $11axy + 13x - 14y$, $2y - 4x$, and $3y + x - 8axy$.
7. Simplify $(5x - 3b) + (2x + b) - (4x - 2b - x + 5b)$.
8. Add $19b + 3c$, $2b - 7c$, $2c - 14b$, and $c + 8b$.
9. Simplify $-(a - 3b - c) - (2c - a - 5b) + (a - c + b)$.
10. Subtract $2x + 4y + z$ from $13x - 3y - 5z + 8$.
11. Solve $5(x - 7) + 3(14 - x) + 60 = 1 - 10x$.
12. Solve $13(1 - x) - 6(2x - 5) = 80 + 12x$.
13. Add $7x - 3y - 4$, $5x + 2y + 5$, and $3y - 8x - 6$.
14. Add $13a + 4b - 9c$, $2c - 8b - 16a$, and $8a - 5b - 8c$.
15. Simplify $8x - [2x + 3(x - 1) - (2x - 3)]$.
16. From $17b - 4a - 2c - 19$ subtract $8c - 5a - 8b + 4$.
17. Simplify $3 - (3 - 2 + 6 + 8 - 3) + 8 - (9 - 3 + 8)$.
18. Solve $3(4 - x) - 2(5 - 6x) = 8x + 4$.
19. Simplify $12 + (2a - 3c - 4b) - (3b - c - a - 8)$.
20. Simplify $5x - (3x - 2 + 2y + x) + 13y - (6 - 3x)$.
21. Add $y - 20$, $4y + 6$, $2y + 4x - 13$, and $2x - 8y - 40$.
22. Subtract $16 - x + 2z - 4y$ from $3x - 5z - 8y$.
23. Solve $19 + (2x - 7) - (31 - 4x - 8 - 2x) = 5x + 7$.
24. Solve $16 + 5x - (8x + 9 - 4x + 17) = 8x - 3$.
25. Solve $6x - 3 - (4x + 8 - 9x) - (5x - 2) = x + 11$.
26. The sum of two numbers is 16. Seven times one is 8 less than 5 times the other. What are the numbers?
27. From a certain number a there is subtracted 3 times the remainder when 8 is subtracted from $2a$. Express the result in terms of a.

REVIEW EXERCISES FOR CHAPTER V

Multiply as indicated:

1. $(x+1)(x-2)$.
2. $(a+b)(2a+b)$.
3. $(2x-y+1)(x-y)$.
4. $(8x-3y)(2x+5y)$.
5. $(a-1+b-c-d)(4a+5b+3c-2d)$.
6. $(4ax-3ay+5az-8)(x+y-z+2)$.
7. $(3a-2b+4c)(2a+3b-c)$.

Divide as indicated:

8. $a^3-12a^2+27a+40$ by $a-5$.
9. $x^5-5x^4y+11x^3y^2-14x^2y^3+9xy^4-2y^5$ by $x^2-3xy+2y^2$
10. $x^4+x^2y^2+y^4$ by x^2-xy+y^2.
11. $a^3+5a^2-2a-24$ by $a^2+7a+12$.
12. $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$ by $a^2-2ab+b^2$.
13. $x^5-5x^3y^2-5x^2y^3+y^5$ by $x^2-3xy+y^2$.
14. Add $12a^2b^2c+8ax$, $6ax-8a^2b^2c$, and $2ax+3a^2b^2c$.
15. Add $5xy^2+3x^2y+4xy$, $2x^2y-6xy^2-3xy$, and $4xy$.
16. Add $6ab-3c-2a$, $2c-4ab-5a$, $5c-a+ab$.
17. From $35ab-8x-9z+13$ subtract $16ab-4z+5x+8$.
18. Subtract $5a-8x-6y$ from $13x+14y-15z-4a$.
19. From $9y-4x-6z-3b$ subtract $8-9y-3x-2z$
20. Solve $(n-4)(6-3n)-(6-n)^2-10=-4n(n-4)$.
21. $(n+2)^2+(n-1)^2+(n+1)^2=3n(n+2)+60n+130$.
22. $2x+4-6(5x-8-7x)+2-4x=6(2-3x)-42$.

23. A man bought a tract of coal land and sold it a month later for $93,840. If his gain was at the rate of 24 % per annum, what did he pay for the land?

24. The melting point of copper is 250 degrees (Centigrade) lower than 4 times that of lead. Ten times the number of degrees at which lead melts minus twice the number at which copper melts equals 1152. Find the melting point of each.

REVIEW EXERCISES FOR CHAPTER VI

Use the formulas for $(a \pm b)^2$ for the following:

1. $[(a+b)+(c-d)]^2$.
2. $[(a+3)-(b+c)]^2$.
3. $[(3a-2b)+5]^2$.
4. $[7x-(4r-s)]^2$.
5. $[(m^2-3)-2(m^3+n)]^2$.
6. $[3(2+y)-2(3+x)]^2$.

Use the formula for $(a+b)(a-b)$, for the following:

7. $[a+b+(c-d)][a+b-(c-d)]$.
8. $[x+y+(u+v)][x+y-(u+v)]$.
9. $[4x-(a-2b)][4x+(a-2b)]$.
10. $[a+2b-(x-y^2)][a+2b+(x-y^2)]$.
11. $(11b^3x-3bx^3)(11b^3x+3bx^3)$.

Find the following as indicated:

12. $(2x-3y+z)^2$.
13. $(2x-3y)^3$.
14. $(x^3+y^3)\div(x+y)$.
15. $(8a^3-125b^3)\div(2a-5b)$.
16. $16c-(41-7c)+(15-8c)$.
17. $-(5a-3c)-(2c-8a)+3a$.
18. $-(-12x-7y-15x)-(-9y+8x+3y)$.
19. $(19x+4y-32x-17x)-12x-(49y+18x-70x)$.

Solve the following equations:

20. $7(m+6)+10m=42-8(2m+2)+181$.
21. $20-3(x-4)+2x=2x+17$.
22. $6(x-3)-2(2-x)=2(x+1)-6$.

23. If two numbers differ by d and if the greater of the numbers is x, how do you represent the other?

24. A father is 3 times as old now as his son was 7 years ago. If the son's age now is represented by x, how is the father's age represented?

25. A picture inside the frame is w inches wide and $w+6$ inches long. The frame is 4 inches wide. Express the area of the frame in terms of w.

REVIEW EXERCISES FOR CHAPTER VII

Factor:

1. x^2+5x+4.
2. x^2-5x+4.
3. a^4-7a^2+12.
4. a^6-a^3-12.
5. $b^2+3b-18$.
6. b^4-b^2-56.
7. a^3+8.
8. $27a^3+b^3$.
9. b^3-27.
10. $8a^3-b^3$.
11. $1+64x^3$.
12. a^6-b^6.
13. $1-64x^3$.
14. w^6+27a^6.
15. w^6-8a^3.
16. $27a^3-8b^3$.
17. x^6+y^6.
18. $125a^3+b^3$.
19. $1+125x^6$.
20. $27x^6-1$.
21. $1-8x^3y^3$.
22. $1+8x^3y^3$.
23. $8x^3+27y^3$.
24. $8x^3-27y^3$.
25. c^4-31c^2+220.
27. $26+39n-22m-33mn$.
26. $ac+d^5a-b^4c-b^4d^5$.
28. $12x^2+11x-56$.
29. $a^2+4ab+4b^2-(a^2-4ab+4b^2)$.
30. $(3x-1)^2-(x^2+4y^2-4xy)$.
31. $(x+3y)^2+(x-2y)^2+2(x+3y)(x-2y)$.
32. $16(a+b)^2-8(a-b)(a+b)+(a-b)^2$.
33. $256x^2-(49x^2+4y^4-28xy^2)$.

Factor the following:

34. $a^2+4ab+4b^2-(x^2-2xy+y^2)$.
35. $(3x-2)^2-(4x^2+9y^2-12xy)$.
36. $x^2+4xy+4y^2-(a^2+2ab+b^2)$.
37. $16x^4y^2-(4x^2+9y^2+12xy.)$
38. $(a+b)^2-(4a^2+9b^2-12ab)$.
39. $a^4+a^2b^2+b^4$.
40. $16x^4+20x^2y^2+9y^4$.

41. The Nile is 100 miles more than twice as long as the Danube. Ten times the length of the Danube minus 4 times the length of the Nile equals 3400 miles. How long is each river?

42. Lead weighs 259 pounds more per cubic foot than cast iron, and 166 pounds more than bronze; while a cubic foot of bronze weighs 807 pounds less than 3 cubic feet of iron. Find the weight per cubic foot of each metal.

REVIEW EXERCISES FOR CHAPTER VIII

Solve the equations:

1. $(x-1)(x+1)(x-3)=0$.
2. $x(x^2-4)(x^2-9)=0$.
3. $x^2+7x+12=0$.
4. $x^3+3x^2-4x-12=0$.
5. $x^3-3x^2-4x+12=0$.
6. $(x-1)(2x-2)+(x-5)^2=(3-x)(24-3x)-7$.
7. $(17x+3)(x-1)+8=(2-x)(6-17x)+19$.

Perform the operations indicated:

8. $(a-2)(6a-4)+2(a-1)^2=(6-a)(30-8a)+4$.
9. $5-(a+b-c-d+8)+(3+a+c-d)-5$.
10. Add $6a+9$, $8a-13$, $46a-8$, and $6-54a$.
11. From $3-4a-5c+8x^2$ subtract $2x^2-2a-4c+8$.
12. $(4ab-6ac-5ad)(b-c+d)$.
13. From $6(a+2)+3(c+4)-2(b-d)$
subtract $2(a+2)-2(c+4)+3(b-d)$.

14. The sum of two numbers is 13 and the sum of their squares is 97. Find the numbers.

15. The sum of two numbers is 38 and their product is 240. Find the numbers.

16. The sum of two numbers is 23 and their product is 120. Find the numbers.

17. The difference between two numbers is 14 and their product is 176. Find the numbers.

18. A rectangular field is 10 rods longer than it is wide, and its area is 1200 square rods. Find its dimensions.

19. An open box is made from a square piece of tin by cutting out a 6-inch square from each corner and turning up the sides. How large is the original square if the box contains 150 cubic inches?

REVIEW EXERCISES FOR CHAPTER IX

Find the H. C. F. of the following:

1. $x^2 - y^2$, $x^3 - y^3$, $x^2 - 2xy + y^2$.
2. $x^2 + 2xy + y^2$, $x^3 + y^3$, $x^2 + xy$.
3. $a^2 + 7a + 12$, $a^2 - 4$, $a^2 - 9$.
4. $125 + m^3$, $25 - m^2$, $m^2 + 10m + 25$.

Find the L. C. M. of the following:

5. $x^2 - 11x + 30$, $x^2 - 36$, $x^2 - 25$.
6. $a^2 - a - 6$, $a^2 - 7a + 12$, $a^2 - 2a - 8$.
7. $a - 3$, $a^2 + 3a + 9$, $a^3 - 27$.
8. $x^2 - 3x + 2$, $x^2 - 5x + 6$, $x^2 - 4x + 3$.

Factor the following:

9. $(2 - x)^2 - 2(2 - x)(x - 1) + (x - 1)^2$.
10. $(2 + y)^2 + 2(2 + y)(1 + x) + (1 + x)^2$.
11. $(3a - 2b)^2 - 10(3a - 2b) + 25$.
12. $(6a - b)^2 + (2a + 1)^2 - 2(6a - b)(2a + 1)$.
13. $25(a + b)^2 + 50(a + b)(a - b) + 25(a - b)^2$.
14. $x^2 + 12x(a + b + c) + 36(a + b + c)^2$.
15. $49(m - 3)^4 + 36(m + 1)^6 - 84(m - 3)^2(m + 1)^3$.
16. $16(x - y)^2 - 16(x - y)(x + y) + 4(x + y)^2$.
17. $-30(a + b)(a - b)^2 + 25(a - b)^4 + 9(a + b)^2$.

18. Express the average of the numbers 3, 8, -9, 12; also of the numbers $3a$, b, $2c$, $-5b$.

19. Express the sum of the squares of four consecutive even integers of which $2n$ is the smallest.

20. Express the sum of the squares of four consecutive odd integers of which $2n + 1$ is the greatest.

21. A wall l feet long and h feet high has three windows each k feet wide and m feet high. By now much does the area of the wall exceed that of the windows?

REVIEW EXERCISES FOR CHAPTER X

Perform the following indicated operations:

1. $\dfrac{x-1}{x-2}+\dfrac{x^2-3}{4-x^2}+\dfrac{x}{2+x}$.

2. $\dfrac{1}{(a-b)(b-c)}+\dfrac{1}{(c-b)(c-a)}-\dfrac{1}{(a-c)(b-a)}$.

3. $\left(a-b+\dfrac{4ab}{a-b}\right)\div\left(\dfrac{(a+b)(b+2a)}{2a^2+3ab+b^2}\right)$.

4. $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}$.

5. $\dfrac{\dfrac{x-y}{x}-\dfrac{x+y}{y}}{\dfrac{x-y}{y}+\dfrac{x+y}{x}}$.

6. $\left\{\dfrac{2}{x}-\dfrac{1}{a+x}+\dfrac{1}{a-x}\right\}\div\left\{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right\}$.

7. $\left(2a-\dfrac{a^2-b^2}{a}\right)\left(3b+\dfrac{a^2+b^2}{b}\right)\div\left(\dfrac{a^2}{b^2}+5+\dfrac{4b^2}{a^2}\right)$.

8. $\dfrac{x-y}{x-2z}-\dfrac{x+y}{x+2z}+\dfrac{y-4z^2}{4z^2-x^2}$.

9. $\left(4-\dfrac{a^2+2ab+b^2}{ab}\right)\left(\dfrac{4ab}{a^2-2ab+b^2}+1\right)$.

10. $\dfrac{a^3+27}{a^3-8}\div\dfrac{a^2b-3ab+9b}{a^2+2a+4}$.

11. $\left(y+\dfrac{xy}{y-x}\right)\left(y-\dfrac{xy}{x+y}\right)\left(\dfrac{y^2-x^2}{y^2+x^2}\right)$.

12. $\left(\dfrac{x-2}{x-3}-\dfrac{x+3}{x+4}\right)\div\left(\dfrac{1}{x+1}+\dfrac{7}{x-3}\right)$.

13. The world's gold production in 1908 was 29 million dollars less than 3 times that of 1893, and the production in 1900 was 59 million less than twice that of 1893. The production of 1900 and 1908 together amounted to 697 million. How much was produced each year?

REVIEW EXERCISES FOR CHAPTER XI

Solve:

1. $\frac{x}{4}+\frac{x}{8}+\frac{x}{16}-\frac{x}{2}+\frac{x}{32}=-2.$

2. $\frac{y-3}{4}-\frac{y+9}{10}=\frac{y-11}{4}-2.$

3. $\frac{y-4}{3}+\frac{y+2}{3}+\frac{y+8}{3}=2y-20.$

4. $\frac{y}{2}+\frac{y+20}{4}+\frac{y+5}{5}=25.$

5. $\frac{y}{3}-\frac{y+20}{5}+\frac{y-5}{5}+\frac{y-10}{2}=15.$

6. $\frac{x}{x-1}-\frac{x}{x+1}=\frac{4}{(x-1)(x+1)}.$

7. $-\frac{x-1}{x+2}+\frac{x+1}{x-3}=\frac{27}{(x+2)(x-3)}.$

8. $\frac{2x-1}{x+2}-\frac{4x-1}{2x-3}=\frac{-10}{(x+2)(2x-3)}.$

9. $\frac{a}{3}+\frac{a+7}{4}-\frac{a-3}{3}=\frac{a+227}{5}-1.$

10. $\frac{a-1}{2}+\frac{a+1}{2}+\frac{a-3}{12}=2+a.$

11. $\frac{a+1}{4}+\frac{a-3}{4}+\frac{a-7}{4}=2a-26.$

12. $\frac{n+1}{3}+\frac{n+3}{4}+\frac{n-1}{4}=\frac{n+13}{3}+\frac{n-2}{3}.$

Perform the following indicated operations:

13. $16ax+4-(8-8ax-a)-(12ax-13-ax).$

14. $a^2b-(3b-8a^2-7)+3ab^2-(4ab^2+8-2a^2).$

15. Add $15ax^2+3bc^2$, $2bc^2-7ax^2$, and $5+2ax^2-5bc^2$.

16. Add $16-7ab-2a^2+5ab$, $4a^2-2ab$, and $5ab-8$.

17. Add $51x^2y-35+12a^2$, $41-17a^2-57x^2y$, and $3x^2y$.

18. How do you represent a fraction whose numerator is 3 less than twice that of $\frac{m}{n}$ and whose denominator is equal to 3 times the sum obtained by adding 2 to the denominator of this fraction?

19. There are two numbers such that if one half their product is divided by twice their sum the result is 12 times their difference. Write an equation representing this relation between the numbers.

REVIEW EXERCISES FOR CHAPTER XII

1. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a+b}{a}=\frac{c+d}{c}$.

2. Which is the greater ratio, $\frac{x+3y}{x+4y}$ or $\frac{x+5y}{x+6y}$, x and y both being positive?

3. Find a fourth proportional to 25, 75, and 100.

Solve:

4. $(9x-3)(4-x)+(x-3)^2=-8(x+2)^2+94$.

5. $(x+1)^2+(x+2)^2+(x+3)^2=(3x-1)(x+12)-43$.

6. $(2x+5)(x-7)-(x-1)^2=(x+1)(x+2)-28$.

7. $3(5-x)^2-(2x-1)(x-1)=(x-7)(x+10)+17x+50$.

8. $(32+x)(4x-1)+(5-x)^2+(x-1)^2=6(x+1)^2+194$.

9. $(2x-7)(5-x)-(2-5x)(1-x)=-x(7x-34)-17$.

10. $\frac{x+8}{2}-\frac{x-9}{12}+\frac{x-17}{6}=\frac{4x-7}{2}+\frac{2x+6}{3}+\frac{5-31x}{12}$.

11. $\frac{3x-1}{6}-\frac{3x+3}{3}+\frac{x-1}{2}=\frac{x+5}{6}+4x-\frac{20}{3}$.

12. $\frac{x-b}{a}+\frac{x+a}{b}=2$.

13. $\frac{3x-16}{2}+\frac{21}{x-8}=\frac{6x-11}{4}$.

14. How may the signs of the factors of a product be changed without changing the value of the product? Make all possible changes of sign in $(a-b)(b-c)(c-d)$ which will not change its value. Also make all possible changes of sign which will change the sign of the product.

15. State how the signs involved in a fraction may be changed without changing the value of the fraction. Make all possible changes of signs which will not change the value of $\frac{a}{b}$; also of $-\frac{a}{b}$.

16. Make three changes of signs each leaving unchanged the value of the expression $\frac{x-y}{(a-b)(b-c)}$.

17. Reduce $\frac{a-b}{x-y}$, $\frac{b-c}{y-z}$, and $\frac{a+b}{(y-x)(z-y)}$ to fractions having a common denominator.

18. Reduce $\frac{a}{(a-b)(b-c)}$, $\frac{b}{(c-d)(b-a)}$, and $\frac{c}{(c-b)(d-c)}$ to fractions having a common denominator.

REVIEW EXERCISES FOR CHAPTER XIII

1. What values of the letters in an identity satisfy it?

2. State in the form of identities as many as possible of the eighteen principles given in this book.

In Examples 3–12, determine which are identities:

3. $(x+y)^2 = x^2 + 2xy + y^2$.
4. $(x-y)^2 = x^2 - 2xy + y^2$.
5. $(x+1)^2 = x^2 - 2$.
6. $a^2 + ab + b^2 = (a+b)^2 - ab$.
7. $a^2 - b^2 = (a-b)(a+b)$.
8. $(a-b)(a^2+ab+b^2) = a^3 - b^3$.
9. $(a+b)(a^2-ab+b^2) = a^3 + b^3$.
10. $(x+b)(x+a) = x^2 + (a+b)x + ab$.
11. $(x-a)(x+b) = x^2 - (a+b)x + ab$.

12. $(a+b-c)^2 = a^2+b^2+c^2+2ab-2ac-2bc.$

Solve the following equations for x:

13. $(x-a)(x-a) = -x(a-x)+(a+b)x.$

14. $\dfrac{a}{b} = \dfrac{x}{c-x}.$

15. $\dfrac{4a}{3x}+\dfrac{4a}{x} = \dfrac{3}{2}+\dfrac{5a}{6x}.$

16. $\dfrac{2a-3b}{3x}+\dfrac{b}{x}+\dfrac{3a}{2x} = \dfrac{13a}{6}.$

17. $(x-a)^2+(x-b)^2 = 2(x-c)^2.$

18. $ax-a(b-x)+ac = 3ab.$

19. $\dfrac{2x-a}{x+b}-\dfrac{2x-b}{2x+a} = 1.$

20. $\dfrac{3}{x+a}+\dfrac{4}{x-b} = \dfrac{7}{x-a}.$

21. $\dfrac{1}{x-b}+\dfrac{1}{x-a} = \dfrac{2}{x-c}.$

22. $\dfrac{7}{x-a}-\dfrac{3}{x+a} = \dfrac{4}{x+b}.$

23. Solve for each letter in terms of the others:

$$\frac{1}{a}+\frac{1}{b} = \frac{1}{c}.$$

24. Solve for a $\quad \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d} = 0.$

25. The sum of two numbers is s and one of them is x. How do you represent the other?

26. The difference between two numbers is d and one of them is x. How do you represent the other?

27. The sum of two numbers is a and their difference is b. Find the numbers in terms of a and b.

28. The difference between two numbers is a and one is twice the other. Find each number in terms of a.

REVIEW EXERCISES FOR CHAPTER XIV

Solve:

1. $\begin{cases} 6y - x = 7 + 4y, \\ 5x + 8y = 1. \end{cases}$

2. $\begin{cases} x - y = 37, \\ 2x + 3y = 314 + 13y. \end{cases}$

3. $\begin{cases} 2x - 3y = y + 6, \\ x - 2y = 4y + 3. \end{cases}$

4. $\begin{cases} 5x + 3y = 0, \\ 2x + y = 1. \end{cases}$

5. $\begin{cases} 2x + 3y = 6x - 1, \\ 3x - 2y = 3. \end{cases}$

6. $\begin{cases} 5x - 3y = 0, \\ 2x + 2 - 6y = 2 - x. \end{cases}$

7. $\begin{cases} \dfrac{5x-3}{4} - \dfrac{2y-5}{2} = \dfrac{1}{2}, \\ \dfrac{3x+5}{2} + \dfrac{y-10}{5} = 6. \end{cases}$

8. $\begin{cases} \dfrac{2x-9}{3} - \dfrac{4y-2}{5} = -1, \\ \dfrac{7x+4}{2} + \dfrac{y+6}{4} = 3. \end{cases}$

9. $\begin{cases} \dfrac{a+7}{3} + \dfrac{4b-1}{6} = \dfrac{1}{2}, \\ \dfrac{4a+7}{3} + \dfrac{7b+3}{5} = -5. \end{cases}$

10. $\begin{cases} x + 2y + 2z = 3, \\ 3x - 4y + z = 19, \\ -2x + 6y + 3z = 0. \end{cases}$

11. $\begin{cases} 2x + 5y + 7z = 7, \\ 3x - 9y - 2z = 23, \\ -x + 3y + 3z = -10. \end{cases}$

12. $\begin{cases} \dfrac{x}{3} + \dfrac{y}{4} + \dfrac{z}{6} = 6, \\ \dfrac{x}{2} + \dfrac{y}{8} - \dfrac{z}{12} = 3, \\ \dfrac{x}{6} - \dfrac{y}{2} + \dfrac{z}{3} = 1. \end{cases}$

13. $\begin{cases} x + y - z = 35, \\ \dfrac{x}{3} + \dfrac{y}{5} + z = 15, \\ \dfrac{x}{5} + \dfrac{y}{5} - z = 3. \end{cases}$

14. If r represents the rate in miles per hour at which a train is moving, how far will it go in t hours? Another train runs 10 miles per hour faster. Express in symbols the sum of the distances which these two trains travel in t hours.

15. If r_1 is the rate of a current and r_2 the rate of a steamer in still water, express the distance which the steamer can go in t hours: (a) with the current; (b) against the current.

REVIEW EXERCISES FOR CHAPTER XV

1. Describe the method of locating a point geographically on the earth's surface.

2. Describe the method for locating a point graphically on squared paper.

3. What is meant by a *linear* equation in two unknowns? Write such an equation.

4. How many points do you need to locate on the graph of a linear equation before you can draw the whole graph?

5. Construct graphs of the equations $3y - 2x = 12$ and $5x + 2y = 8$.

6. What is the relation between the graphs of the equations $x + 3y = -6$ and $2x - 4y = -12$?

7. What is meant by *independent* equations? by *simultaneous* equations?

8. Write a pair of equations which are not independent. Construct their graphs.

9. Write a pair of equations which are not simultaneous Construct their graphs.

Solve the following equations:

10. $\dfrac{x}{x-2} + \dfrac{4x}{x+3} = 5.$

11. $\dfrac{x+a}{a+b} + \dfrac{x+b}{a-b} = 1.$

12. $\begin{cases} x + 2y = 4, \\ 2x + y = -1. \end{cases}$

13 $\begin{cases} 5x + 9y = 19, \\ 3x - y = 5. \end{cases}$

14. $\begin{cases} 3x - 7y = -11, \\ 2x + y = 4. \end{cases}$

15. $\begin{cases} 5x - 3y = 4 - 2x + 7y, \\ 5y + x = 7. \end{cases}$

16. $\begin{cases} x - 3y + z = 10, \\ 2x + y - z = 1, \\ 3x - 2y + 5z = 31. \end{cases}$

17. $\begin{cases} x + y + z = 1, \\ 3x + 4y - z = 1, \\ -2x - y + 3z = 5. \end{cases}$

REVIEW EXERCISES FOR CHAPTER XVI

Find the square roots of:

1. $x^6 + 2x^4 + 2x^3 + x^2 + 2x + 1$.
2. $x^8 + 2x^6 + 3x^4 + 2x^2 + 1$.
3. $x^6 + 4x^5 + 10x^4 + 16x^3 + 17x^2 + 12x + 4$.
4. $4a^4 - 12a^3 - 7a^2 + 24a + 16$.
5. $a^2 + \frac{2a}{b} + \frac{2a}{c} + \frac{2}{bc} + \frac{1}{b^2} + \frac{1}{c^2}$.
6. $\frac{1}{4}a^4 - \frac{1}{3}a^2b^3 + \frac{1}{9}b^6 + \frac{1}{4}a^2c^4 + \frac{1}{16}c^8 - \frac{1}{6}b^3c^4$.
7. $\frac{4a^4}{b^4} - \frac{4a^3}{b^3} - \frac{13a^2}{3b^2} + \frac{8a}{3b} + \frac{16}{9}$.
8. $\frac{x^2}{y^2} + \frac{2x}{z} + \frac{y^2}{z^2} - \frac{2y}{x} + \frac{z^2}{x^2} - \frac{2z}{y}$.

Approximate the square roots of each of the following to two places of decimals:

9. 7.9482.
10. 4578.9.
11. 390.07.
12. 9.176.
13. .0048.
14. .04791.

Simplify the following:

15. $\sqrt{18}$; $\sqrt{20}$; $\sqrt{a^2b^3}$; $\sqrt{a^2b^2x}$.
16. $\sqrt{\frac{2}{3}}$; $\sqrt{\frac{2}{5}}$; $\sqrt{\frac{3}{a}}$; $\sqrt{\frac{1}{6}}$.
17. $\sqrt{45} + \sqrt{\frac{1}{5}}$; $\sqrt{48} + \sqrt{75} - \sqrt{3}$.
18. Divide $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ by $x + y$.
19. Divide $x^6 - y^6$ by $x^2 + xy + y^2$.
20. Divide $x^8 - y^8$ by $x^3 + x^2y + xy^2 + y^3$.

21. The older of two sisters is now 8 years less than twice as old as the other. If x represents the age of the younger sister, represent in symbols twice the sum of their ages 7 years ago.

22. A rear wheel of a wagon has a circumference 4 feet greater than that of a front wheel. If the circumference of the rear wheel is x feet, represent in symbols the number of revolutions each wheel must make to go one mile.

REVIEW EXERCISES FOR CHAPTER XVII

Simplify the following:

1. $\sqrt{32}+\sqrt{72}-\sqrt{50}$.
2. $2a\sqrt{a^3b}-b\sqrt{b^3a}+\sqrt{ab}$.
3. $\sqrt{a^3+2a^2b+ab^2}+\sqrt{a^3-2a^2b+ab^2}-2\sqrt{a^3}$.
4. $\sqrt{a^3-a^2b}-\sqrt{b^2a-b^3}+\sqrt{a^3b^2-a^2b^3}$.
5. $\sqrt{(x+y)(x^2-y^2)}-\sqrt{(x+y)^2(x-y)}-\sqrt{x^3-x^2y}$.

Solve the following equations:

6. $\sqrt{x^2-8}+x=8$.
7. $\sqrt{5a-24}+4=\sqrt{5a}$
8. $\sqrt{2a-1}=7-\sqrt{2a+6}$.
9. $\sqrt{x+2}=\sqrt{x-6}+2\sqrt{x-5}$.
10. $\sqrt{2x^2-2}=x+1$.
11. $\sqrt{2x+7}=\sqrt{x}+2$.
12. $\dfrac{\sqrt{a}}{\sqrt{x-a}}-\dfrac{\sqrt{x+2a}}{2\sqrt{a}}=0$.
13. $\dfrac{\sqrt{5x+1}}{\sqrt{2x-5}}=\sqrt{\dfrac{9x+1}{3x-5}}$.
14. $\sqrt{x+2}=3-\sqrt{3-x}$.
15. $\dfrac{9x+8}{\sqrt{8x+1}}=\sqrt{10x+19}$

* Rationalize the denominators of the following:

16. $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$.
17. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$.
18. $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$.
19. $\dfrac{\sqrt{x-a}+b}{\sqrt{x+a}-c}$.
20. $\dfrac{3}{\sqrt{7}-\sqrt{4}}$.
21. $\dfrac{\sqrt{7}+\sqrt{2}}{\sqrt{7}-\sqrt{2}}$.
22. $\dfrac{a+b}{\sqrt{a}+\sqrt{b}}$.
23. $\dfrac{2+\sqrt{3}}{4-\sqrt{3}}$.
24. $\dfrac{\sqrt{8}+7}{\sqrt{8}-7}$.
25. $\dfrac{\frac{1}{3}\sqrt{2}-1}{\frac{1}{3}\sqrt{2}+1}$.

26. A and B working together can do a piece of work in 12 days. B and C working together can do it in 13 days, and A and C working together can do it in 10 days. How long will it require each to do it when working alone?

REVIEW EXERCISES FOR CHAPTER XVIII

Complete the square in each of the following:

1. $4a^2 + 8a$.
2. $9a^2 + 30a$.
3. $x^2 + 3x$.
4. $25x^2 - 7x$.
5. $7b^2 - 3b$.
6. $16b^2 - 7b$.
7. $3x^2 - 4x$.
8. $7x^2 - 11x$.
9. $8x^2 + 7x$.

Solve and check, using § 204:

10. $x^2 - 7x + 9 = 0$.
11. $2x^2 - 5x + 2 = 0$.
12. $7x^2 + 18x - 3 = 0$.
13. $x^2 - 12x + 16 = 0$.
14. $x^2 + 2bx = 3c$.
15. $2x^2 - 5ax = a^2$.
16. $ax^2 + 2bx = 3c$.
17. $4a^2x^2 + 6bx = 3b^2$.
18. $x^2 - 18x + 4 = 0$.
19. $x^2 - 3ax + b = 0$.
20. $x^2 + 9bx + c = 0$.
21. $2x^2 - 7x = 5$.
22. $3ax^2 - 7bx = 3$.
23. $4a^2x^2 - 2bax = ca$.

Reduce the solution of each of the following to simplest form without approximating any roots:

24. $7x^2 = 27$.
25. $5x^2 = 108$.
26. $2x^2 = 3$.
27. $3a^2 = 343$.
28. $5x^2 = 8a^3b$.
29. $2x^2 = 27a(a + b)^2$.
30. $7ax^2 = 98$.
31. $2(a + b)x^2 = 300$.
32. $x^2 = \dfrac{162}{125ab}$.
33. $x^2 = \dfrac{128}{72cd^3}$.
34. $(a - b)x^2 = a + b$.

35. The circumference of the rear wheel of a carriage is 1.8 feet more than that of the front wheel. In running one mile the front wheel makes 48 revolutions more than the rear wheel. Find the circumference of each wheel.

36. The circumference of the rear wheel of a carriage is 1 foot more than that of the front wheel. In going one mile the two wheels together make 920 revolutions. Find the circumference of each.

REVIEW EXERCISES FOR CHAPTER XIX

Solve the following systems of equations:

1. $\begin{cases} x^2 + y^2 = 25, \\ x - y = 4. \end{cases}$

2. $\begin{cases} 2x^2 - 3xy = 12, \\ x + 2y = 4. \end{cases}$

3. $\begin{cases} x^2 - y^2 = 8, \\ x - 2y = 3. \end{cases}$

4. $\begin{cases} xy - y^2 = 14, \\ x + y = 4. \end{cases}$

5. $\begin{cases} x^2 - xy = 8, \\ x + y = -2. \end{cases}$

6. $\begin{cases} 2x^2 - 3y^2 = 7, \\ x - y = 5. \end{cases}$

7.* $\begin{cases} x^3 - y^3 = 56, \\ x - y = 2. \end{cases}$

8.* $\begin{cases} x^4 - y^4 = 63, \\ x^2 + y^2 = 7. \end{cases}$

9.* $\begin{cases} x^2 - xy + 2y^2 = 16, \\ x^2 - 4xy + 4y^2 = 0. \end{cases}$

10.* $\begin{cases} x^2 - xy = 3, \\ y^2 + 3xy = 22. \end{cases}$

11. Divide $x^5 - 3x^4 - 18x^3 + 24x^2 + 52x - 21$ by $x^2 + x - 7$

12. Divide $6a^5 + 5a^4 - 60a^3 + 4a^2 + 71a + 28$ by $3a^2 - 5a - 4$.

Find the square roots of:

13. $16x^8 - 40x^6y^3 + x^4y^6 + 30x^2y^9 + 9y^{12}$.

14. $4m^6 - 20m^5 + 9m^4 + 52m^3 - 14m^2 - 24m + 9$.

15. Two square pieces of land require together 360 rods of fence. If the difference in the area of the pieces is 900 square rods, how large is each piece?

16.* The difference of the cubes of two numbers is 218 and the difference of the numbers is 2. Find the numbers.

17. The hind wheel of a wagon makes 12 revolutions less than the fore wheel in going 720 feet. If the circumference of each wheel were three feet greater, the hind wheel would make 8 revolutions less than the fore wheel in going the same distance as before. Find the circumference of each wheel.

REVIEW EXERCISES FOR CHAPTER XX

Expand by the binomial formula:

1. $(3x-2y)^3$.
2. $(2x+y)^5$.
3. $\left(x+\frac{1}{x}\right)^6$.
4. $(2x+3y)^4$.
5. $(a+b)^7$.
6. $(x-y)^8$.
7. Find the 6th term of $(x+y)^{14}$.
8. Find the 8th term of $(2a-b)^{10}$.
9. Find the 7th term of $(a-3b)^8$.
10. Find the 13th term of $(2a-c)^{15}$.
11. What is the value of x^2-3x if $x=1+\sqrt{5}$? If $x=1-\sqrt{5}$?
12. What is the value of $3x^2-5x+6$ if $x=\frac{2-\sqrt{3}}{2}$? If $x=\frac{2+\sqrt{3}}{2}$?
13. What is the value of $5x^2+7x$ if $x=\frac{a+\sqrt{b}}{2}$? If $x=\frac{a-\sqrt{b}}{2}$?

Find the square root of each of the following:

14. $25a^2+c^2+9b^2-10ac+30ab-6bc$.
15. $x^2+y^2+4z^2+4v^2-2xy+4xz-4xv-4yz+4yv-8zv$.
16. $x^6-2x^5-x^4+3x^2+2x+1$.
17. $x^4-6x^3+13x^2-12x+4$.
18. Divide $6x^4+x^3+12x^2+8$ by $2x^2-x+2$.
19. Divide $3x^6+4x^5-x^4+6x^3-12x^2+8x-12$ by $3x^2-2x+3$.
20. Divide $3a^7-5a^5+8a^3+2a^2-18a+12$ by a^3-a+2.
21. Divide $6a^4+10a^3-9a^2+11a-6$ by $2a^2+4a-3$.
22. A picture inside the frame is w inches wide and l inches long. The frame is a inches wide. Express the area of the frame in terms of a, w, and l.

REVIEW EXERCISES FOR CHAPTER XXI

1. If $y \propto x$ and if $y = 9$ when $x = 17$, find y when $x = 12$.

2. If $y = \frac{k}{x}$ and if $y = 4$ when $x = \frac{1}{3}$, find y when $x = \frac{3}{5}$.

3. If $y \propto x^2$ and if $y = 7$ when $x = 1$, find y when $x = 4$.

4. If $y \propto x^2$ and if $y = 3$ when $x = 3$, find y when $x = 7$.

5. If $y \propto x^2$ and if $y = 6$ when $x = 4$, find y when $x = 10$.

6. If a certain sum of money yields \$ 350 interest in 8 months, how much will the same sum yield in $17\frac{1}{2}$ months?

7. If \$ 2500 yields \$ 680 interest during a certain time, how much money will be required to yield \$ 390 during the same time?

8. If 60 men can do a certain piece of work in 45 days, how long will it take 35 men to do it?

9. Over what distance will a falling body pass in 5 seconds, starting from rest?

10. Over what distance will a body fall in 10 seconds, if it starts with a velocity of 15 feet per second?

11. If a circular disk weighs 16 lbs., how much will a disk weigh if its diameter is 3 times that of the first disk and if it is of the same material and thickness?

12. $$\frac{\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}}{\frac{(a-b-c)(a+b+c)}{ab}}.$$

13. $$\frac{\left(a-\frac{3}{a}\right)^2+16}{\left(a+\frac{3}{a}\right)^2+4}.$$

14. $$\frac{\frac{a}{b^2}}{1+\frac{a^2}{b^2}}+\frac{a}{\frac{(b-a)^2}{b+a}}.$$

15. $$\frac{2+\frac{1}{x-2}+x}{x+\frac{1}{x+2}-2}.$$

MISCELLANEOUS REVIEW EXERCISES

Solve the following:

1. $(x-2)^2-(x-1)(x+2)=6-5x.$
2. $(x-2)(x+2)+(3x-1)(2-x)=(x-2)(5-2x).$
3. $(x-3)^2+(2x+5)^2=(5x-3)(x+5)-7.$
4. $(2-x)^2-(2x-1)^2=(-3x+1)(4+x)-4.$
5. $\begin{cases} 2x-3y-1=0, \\ 5x+2y=12. \end{cases}$
6. $\begin{cases} x+y=a, \\ x-y=b. \end{cases}$
7. $\begin{cases} ax+by=c, \\ ax-by=d. \end{cases}$
8. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=c, \\ \dfrac{x}{a}-\dfrac{y}{b}=d. \end{cases}$
9. $\begin{cases} 3ax-by=2, \\ 2x+3by=6. \end{cases}$
10. $\begin{cases} x-y-3z=-6, \\ 2x+y-z=11, \\ -x+3y+z=16. \end{cases}$
11. $\begin{cases} 2x-3by=c, \\ 2ax-5y=d. \end{cases}$
12. $\begin{cases} 2x-y+3z=20, \\ x+4y-z=-2, \\ 5x+y-6z=6. \end{cases}$
13. $(x-1)^2-(x-8)(2x-1)=-x^2+98.$
14. $(7+x)(x-4)+(1-x)^2=-23+2x^2.$
15. $(12-4x)(2-x)-4(1+x)^2=5x+119.$
16. $(x-17)(59-2x)-(1-x)^2=(6-3x)(x-2)+384.$
17. $(3x-2)+(x-1)^2+(x-2)^2=2(x-1)(x-2)+5.$
18. $(6-3x)(2+x)+16(x-1)^2=13(x+4)^2+364.$
19. $7x+(8x+4)\div 2=4x+9.$
20. $6x+4(4x+2)=85-3(2x+7).$
21. $8+7(6+6n)+2n=2(4n+5)+18n+49.$
22. $5(9x+3)+6x=24x-4(3x+2)+36.$

23. Find the average of the following temperatures: 7 A.M., $-4°$; 8 A.M., $-2°$; 9 A.M., $-1°$; 10 A.M., $+1°$; 11 A.M., $+5°$; 12 M., $+7°$.

Solve the following:

24. $\begin{cases} 3x-7by=2, \\ 2ax+2y=4. \end{cases}$

25. $\begin{cases} 6x+3y=1, \\ 5ax-2by=c. \end{cases}$

26. $\dfrac{7(12x+8)}{4}+13+5x-6=47.$

27. $\dfrac{x+2}{a}=\dfrac{x-1}{b}+\dfrac{3x+2}{ab}.$

28. $\begin{cases} \dfrac{7x-3}{2}-\dfrac{ay+1}{3}=1, \\ \dfrac{bx+5}{4}+\dfrac{y-3}{3}=6. \end{cases}$

29. $\begin{cases} \dfrac{2by}{3a}-3x=2, \\ 5y+\dfrac{2ax}{b}=5. \end{cases}$

30. $\begin{cases} 2x-y-z=8, \\ 3x+2y+z=24, \\ -x-3y+5z=16. \end{cases}$

31. $\begin{cases} 8x-3y-z=0, \\ 2x+2y+3z=10, \\ -x+y+6z=8. \end{cases}$

32. $\dfrac{a-b}{2x-1}=\dfrac{4c}{(2x+1)(2x-1)}-\dfrac{a-b}{2x+1}.$

33. $(1-3x)^2+(2x+1)^2=5x^2+(2x+6)(4x+27).$

34. $(1+2x)(2-3x)+(x-4)(x+4)=(x+14)(18-5x)-1.$

35. A picture inside the frame is 12 inches long and 8 inches wide. If the frame is a inches wide, express its area in terms of a.

36. If h is the length of the hypotenuse of a right triangle and a the length of one side, express the length of the third side in terms of h and a.

37. A rectangular piece of tin is w inches wide and l inches long. If a square a inches on a side is cut out of each corner, express in terms of w, l, and a the volume of a box formed by turning up the sides.

38. A farmer plows a strip a rods wide around a rectangular field w rods wide and l rods long. Express in terms of w, l, and a the area plowed.

Solve the following:

39. $2(x+5)(x-5) = (x-5)(x+1) + (x-2)^2 - 1.$

40. $\begin{cases} \dfrac{x-1}{2} + \dfrac{y+2}{3} = 3\frac{1}{6}, \\ \dfrac{x+1}{3} - \dfrac{y-1}{4} = \frac{7}{6}. \end{cases}$

41. $\begin{cases} \dfrac{x-a}{2} + \dfrac{y-b}{3} = 1, \\ \dfrac{x+a}{3} + \dfrac{y+b}{2} = 1. \end{cases}$

42. $\begin{cases} 2ax - 3by = c, \\ 2ax + 3by = d. \end{cases}$

43. $\begin{cases} x + y + z = 6, \\ 2x - y + z = 3, \\ 3x + 2y - 2z = 1. \end{cases}$

44. $\begin{cases} x - y + z = 4, \\ x - 2y + 4z = 6, \\ 2x + y - 3z = 10. \end{cases}$

45. $\begin{cases} x + y + z = 0, \\ 2x - 4y + z = -3, \\ 3x + 2y + 4z = 3. \end{cases}$

46. $\begin{cases} x + ay = c, \\ bx + y = d. \end{cases}$

47. $\begin{cases} ax + 3y = 2c, \\ bx - 2y = 3d. \end{cases}$

48. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = a, \\ \dfrac{1}{y} + \dfrac{1}{z} = b, \\ \dfrac{1}{z} + \dfrac{1}{x} = c. \end{cases}$

49. $\begin{cases} ax + by = 1, \\ cx + dy = 4. \end{cases}$

50. $\begin{cases} ax - by = c, \\ cx + dy = e. \end{cases}$

51. $\begin{cases} x + y + z = a, \\ 2x - 2y + 2z = b, \\ 3x - y - z = c. \end{cases}$

52. $\begin{cases} ax - y + bz = a, \\ x + ay - z = 1, \\ bx - y + az = b. \end{cases}$

53. If w and l are the length and width of a rectangle, express in symbols the length of its diagonal.

54. The lengths of the two sides of a right triangle are 16 and 24 respectively. Express the length of the hypotenuse in the simplest form without approximating a square root.

55. A father is now twice the age of his son. If x represents the son's age now, express twice the sum of their ages 5 years ago.

56. One number is 3 less than 4 times another. Express one third the sum of the numbers if x represents one of them.

57. If h is the digit in hundreds' place, t the digit in tens' place, and u the digit in units' place of a certain number, express the number obtained by inverting the order of the digits of the given number.

58. h, t, and u are the digits in hundreds', tens', and units' places of a number. Express the number obtained by increasing each digit by 2.

59. $\dfrac{12(5+4x)}{6}-\dfrac{5(6+4x)}{2}+50=x+18.$

60. $15+\dfrac{21(3+x)}{7}+\dfrac{2(6+18x)}{3}=\dfrac{3(9x+12)}{3}+28.$

61. $\dfrac{11(5x+25)}{5}+\dfrac{3(6x-2)}{2}=\dfrac{7(4x+8)}{4}+\dfrac{12x+36}{3}+35.$

62. $\dfrac{2(x+1)}{x-1}-3=-\dfrac{x-1}{x+1}.$

63. $\dfrac{x+a}{x-a}=5-\dfrac{4(x-a)}{x+a}.$

64. $\begin{cases} \dfrac{2a+5}{3}+\dfrac{3b-10}{4}=0, \\ \dfrac{3a+6}{8}-\dfrac{5-4b}{7}=2. \end{cases}$

PROBLEMS ON MOTION

1. A sparrow flies 135 feet per second and a hawk 149 feet per second. The hawk in pursuing the sparrow passes a certain point 7 seconds after the sparrow. In how many seconds from this time does the hawk overtake the sparrow?

2. A courier starts from a certain point, traveling r_1 miles per hour, and a hours later a second courier starts, going at the rate of r_2 miles per hour. In how long a time will the second overtake the first, supposing r_2 to be greater than r_1?

If the second courier requires t hours to overtake the first, the latter had been on the way $t+a$ hours. Thus the distance covered by the second courier is r_2t and by the first $r_1(t+a)$. As these numbers are equal we have

$$r_2t=r_1(t+a).$$

This formula summarizes the solution of all problems like 3 and 4 on the next page.

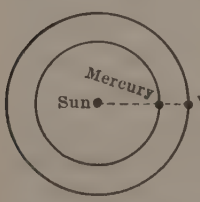
3. In an automobile race A drives his machine at an average rate of 53 miles per hour, while B, who starts $\frac{1}{4}$ hour later, averages 57 miles per hour. How long does it require B to overtake A? Use the formula on page 343.

4. A freight steamer leaves New York for Liverpool, averaging $10\frac{1}{2}$ knots per hour, and is followed 4 days later by an ocean greyhound, averaging $25\frac{1}{2}$ knots per hour. In how long a time will the latter overtake the former?

5. One athlete makes a lap on an oval track in 26 seconds, another in 28 seconds. If they start together in the same direction, how soon will the first gain one lap? Two laps?

Let one lap be the unit of distance. Since the first covers one lap in 26 seconds, his rate per second is $\frac{1}{26}$. Likewise the rate of the other is $\frac{1}{28}$. If t is the required number of seconds, the distance covered by the first is $\frac{1}{26}t$ and by the second $\frac{1}{28}t$. If the first goes one lap farther than the second, the equation is $\frac{1}{26}t = \frac{1}{28}t + 1$.

6. Two automobiles are racing on a circular track. One makes the circuit in 31 minutes and the other in $38\frac{1}{2}$ minutes. In what time will the faster machine gain 1 lap on the slower?



7. The planet Mercury makes a circuit around the sun in 3 months and Venus in $7\frac{1}{2}$ months. Starting in conjunction, as in the figure, how long before they will again be in this position?

Note that the problem may be solved just as if the two planets were moving in the same orbit at different rates.

8. Saturn goes around the sun in 29 years and Jupiter in 12 years. Find the time between two conjunctions.

9. Uranus makes the circuit of its orbit in 84 years and Neptune in 164 years. If they start in conjunction, how long before they will be in conjunction again? *Ans.* $172\frac{1}{5}$ years.

10. The hour hand of a watch makes one revolution in 12 hours and the minute hand in 1 hour. How long is it from the time when the hands are together until they are again together?

11. One object makes a complete circuit in a units of time and another in b units (of the same kind). In how many units of time will one overtake the other, supposing b to be greater than a?

The solution of this problem summarizes the solution of all problems like those from 5 to 10.

12. At what times between 12 o'clock and 6 o'clock are the hands of a watch together? (Find the time required to gain one circuit, two circuits, etc.)

PROBLEMS INVOLVING THE LEVER

1. A teeter board is in balance when two boys, A and B, weighing 105 and 75 pounds respectively, are seated at distances 5 and 7 feet from the fulcrum, because $7 \cdot 75 = 5 \cdot 105$. If now two boys weighing 48 and 64 pounds are seated on the same board with the other boys, the teeter will again be in balance if their distances are 4 and 3 feet, because

$$7 \cdot 75 + 4 \cdot 48 = 5 \cdot 105 + 3 \cdot 64$$

The weight of the boy multiplied by his distance from the fulcrum is called his **leverage**. The sum of the leverages on the two sides must be the same. Hence, if the teeter balances when two boys, weighing respectively w_1 and w_2 pounds, are at distances d_1 and d_2 on one side, and two boys, weighing w_3 and w_4 pounds, are at distances d_3, d_4 on the other side, then

$$w_1 d_1 + w_2 d_2 = w_3 d_3 + w_4 d_4.$$

2. If two boys weighing 75 and 90 pounds sit at distances of 3 and 5 feet respectively on one side and one weighing 82 pounds sits at 3 feet on the other side, where should a boy weighing 100 pounds sit to make the board balance?

3. A beam carries a weight of 240 pounds $7\frac{1}{2}$ feet from the fulcrum and a weight of 265 pounds at the opposite end which is 10 feet from the fulcrum. On which side and how far from the fulcrum should a weight of 170 pounds be placed so as to make the beam balance?

4. Two boys, A and B, having a 50-lb. weight and a teeter board, proceed to determine their respective weights as follows: They find that they balance when B is 9 feet and A is 7 feet from the fulcrum. If B places the 50-lb. weight on the board beside him, they balance when B is 3 and A is 4 feet from the fulcrum. How heavy is each boy?

5. C is $6\frac{1}{2}$ feet from the point of support and balances D, who is at an unknown distance from this point. C places a $33\frac{1}{3}$-lb. weight beside himself on the board and, when $4\frac{2}{3}$ feet from the fulcrum, balances D, who remains at the same point as before. D's weight is 84 pounds. What is C's weight, and how far is D from the fulcrum?

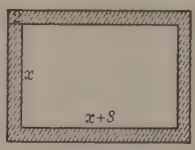
6. E weighs 95 pounds and F 110 pounds. They balance at certain unknown distances from the fulcrum. E then takes a 30-pound weight on the board, which compels F to move 3 feet farther from the fulcrum. How far from the fulcrum was each of the boys at first?

7. A who weighs 60 pounds sits 3 feet from the fulcrum. B who weighs 90 pounds sits 4 feet from the fulcrum on the same side as A. How far from the fulcrum on the opposite side must C, who weighs 108 pounds, sit in order to make the teeter board balance?

PROBLEMS INVOLVING GEOMETRY

1. A picture is 4 inches longer than it is wide. Another picture, which is 12 inches longer and 6 inches narrower, contains the same number of square inches. Find the dimensions of the pictures.

2. A picture, not including the frame, is 8 inches longer than it is wide. The area of the frame, which is 2 inches wide, is 176 square inches. Find the dimensions of the picture.



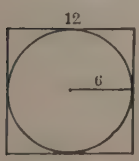
3. A picture, including the frame, is 10 inches longer than it is wide. The area of the frame, which is 3 inches wide, is 192 square inches. What are the dimensions of the picture?

4. The base of a triangle is 11 inches greater than its altitude. If the altitude and the base are both decreased 7 inches, the area is decreased 119 square inches. Find the base and the altitude of the triangle.

5. The base of a triangle is 3 inches less than its altitude. If the altitude and the base are both increased by 5 inches, the area is increased by 155 square inches. Find the base and the altitude of the triangle.

6. A square is inscribed in a circle and another circumscribed about it. The area of the strip inclosed by the two squares is 25 square inches. Find the radius of the circle.

7. Find the sum of the areas of a circle of radius 6 and the square circumscribed about the circle.



The area of the circle is $6^2\pi = 36\pi$, and the area of the square $4 \cdot 6^2 = 4 \cdot 36$; *i.e.* the square contains 4 squares whose sides are 6. The sum of the areas is

$$4 \cdot 36 + 36\pi = (4 + \pi)36 = (4 + 3\tfrac{1}{7})\,36.$$

8. Find an expression for the sum of the areas of a circle of radius r and the circumscribed square. (Solve Example 7 by substituting in the formula here obtained.)

9. If the sum of the areas of a circle and the circumscribed square is 64, find the radius of the circle.

By the formula obtained under Example 8,

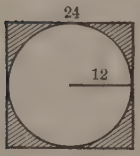
$$64 = (4 + \pi)\, r^2 = \tfrac{50}{7} r^2.$$

Hence,

$$r = \sqrt{8.96} = 2.99.$$

10. If the sum of the areas of a circle and the circumscribed square is 640 square feet, find the radius of the circle.

11. The sum of the areas of a circle and the circumscribed square is a. Find an expression representing the radius of the circle. (Replace π by $3\frac{1}{7}$ before simplifying.)



12. If the radius of a circle is 12, find the difference between the areas of the circle and the circumscribed square.

13. If the radius of a circle is r, find the difference between the areas of the circle and the circumscribed square. (Solve Example 12 by the use of the formula obtained here.)

14. If the radius of a circle is 16, find the area of the inscribed square. (This is the same problem as finding the area of a square whose diagonal is 32.)

15. If the radius of a circle is r, find an expression representing the area of the inscribed square.

16. If the radius of a circle is 12, find the difference between the area of the circle and the area of the inscribed square.

17. If the radius of a circle is r, find an expression representing the difference between the areas of the circle and of the inscribed square.

18. The radius of a circle is 10. Find the area of an inscribed hexagon.

19. The radius of a circle is 6. Find the difference between the areas of the circle and the inscribed hexagon.

20. Find an expression representing the difference between the areas of a circle with radius r and the inscribed regular hexagon.

MISCELLANEOUS PROBLEMS

1. Divide the number 645 into two parts, such that 13 times the first part is 20 more than 6 times the other.

2. Divide the number a into two parts, such that b times the first part is c more than d times the second part.

3. The sum of three numbers is 98. The second is 7 greater than the first, and the third is 9 greater than the second. What are the numbers?

4. The sum of three numbers is s. The second is a greater than the first, and the third is b greater than the second. What are the numbers?

5. One boy runs around a circular track in 26 seconds, and another in 30 seconds. In how many seconds will they again be together, if they start at the same time and place and run in the same direction?

6. A bird flying with the wind goes 65 miles per hour, and flying against a wind twice as strong it goes 20 miles per hour. What is the rate of the wind in each case?

7. A steamer going with the tide makes 19 miles per hour, and going against a current $\frac{1}{2}$ as strong it makes 13 miles per hour. What is the speed of the steamer in still water?

8. Find the time between 4 and 5 o'clock when the hands of the clock are 30 minute spaces apart.

9. A man takes out a life insurance policy for which he pays in a single payment. Thirteen years later he dies and the company pays $ 12,600 to his estate. It was found that his investment yielded 2 % simple interest. How much did he pay for the policy?

10. After deducting a commission of 3 % for selling bonds, a broker forwarded $ 824.50 What was the selling price of the bonds?

11. A broker sold stocks for $ 1728 and remitted $ 1693.44 to his employer. What was the rate of his commission?

12. The difference between the areas of a circle and its circumscribed square is 12 square inches. Find the radius of the circle. (See problem 11, page 348.)

13. The difference between the areas of a circle and its inscribed square is 12 square inches. Find the radius of the circle.

14. The difference between the areas of a circle and the regular inscribed hexagon is 12 square inches. Find the radius of the circle.

15. The altitude of an equilateral triangle is 6. Find its side and also its area. Find the side and area if the altitude is h.

16. The radius of a circle is 3 feet. Find the area of the regular circumscribed hexagon. Find the area if the radius is r feet.

17. The radius of a circle is r. Find the difference between the areas of the circle and the regular circumscribed hexagon.

18. The difference between the areas of a circle and the regular circumscribed hexagon is 9 square inches. Find the radius of the circle.

19. A circle is inscribed in a square and another is circumscribed about it. The area of the ring formed by the two circles is 25 square inches. How long is the side of the square?

20. In a building there are at work 18 carpenters, 7 plumbers, 13 plasterers, and 6 hod carriers. Each plasterer gets \$1.90 per day more than the hod carriers, the carpenters get 35 cents per day more than the plasterers, and the plumbers 50 cents per day more than the carpenters. If one day's wages of all the men amount to \$183.45, how much does each get per day?

21. A train running 46 miles per hour leaves Chicago for New York at 7 A.M. Another train on the same road running 56 miles per hour leaves at 9.30 A.M. Find when the trains will be 15 miles apart. (Two answers.)

22. There is a number consisting of three digits, those in tens' and units' places being the same. The digit in hundreds' place is 4 times that in units' place. If the order of the digits is reversed, the number is decreased by 594. What is the number?

23. A hound pursuing a deer gains 400 yards in 25 minutes. If the deer runs 1300 yards a minute, how fast does the hound run? If the hound gains v_1 yards in t minutes, and the deer runs v_2 yards per minute, find the speed of the hound.

24. The altitude of Popocatepetl is 1716 feet less than that of Mt. Logan, and the altitude of Mt. St. Elias is 316 feet greater than that of Popocatepetl. Find the altitude of each mountain, the sum of their altitudes being 55,384 feet.

25. It is 4 times as far from New York City to Cincinnati as from New York to Baltimore. Twice the distance from New York to Cincinnati minus 5 times that from New York to Baltimore equals 567 miles. How far is it from New York to each of the other cities?

26. A disabled steamer 240 knots from port is making only 4 knots an hour. By wireless telegraphy she signals a tug, which comes out to meet her at 17 knots an hour. In how long a time will they meet? If the steamer is s knots from port and is making v_1 knots per hour, and if the tug makes v_2 knots per hour, find how long before they will meet.

27. A motor boat, going 11 miles per hour, starts $7\frac{2}{3}$ miles behind a sailboat going $6\frac{1}{2}$ miles per hour. How far apart will they be in $1\frac{1}{3}$ hours? If the motor boat starts s miles behind the sailboat and runs v_1 miles per hour, while the sailboat runs v_2 miles per hour, how far apart will they be in t hours?

28. An ocean liner making 21 knots an hour leaves port when a freight boat making 8 knots an hour is already 1240 knots out. In how long a time will the two boats be 280 knots apart? Is there more than one such position? If the liner makes v_1 knots per hour and the freight boat, which is s_1 knots out, makes v_2 knots per hour, how long before they will be s_2 knots apart?

29. A passenger train running 45 miles per hour leaves one terminal of a railroad at the same time that a freight running 18 miles per hour leaves the other. If the distance is 500 miles, in how many hours will they meet? If they meet in 8 hours, how long is the road? If the rates of the trains are v_1 and v_2 and the road is s miles long, find the time.

30. The melting temperature of glass is 276 degrees (Centigrade) higher than twice that of zinc. One half the number of degrees at which glass melts plus 7 times the number at which zinc melts equals 3434. Find the melting point of each.

31. The melting temperature of nickel is 496 degrees (Centigrade) higher than that of silver. Three times the number of degrees at which nickel melts plus 2 times the number at which silver melts equals 6258. Find the melting point of each.

HARVARD UNIVERSITY

ELEMENTARY ALGEBRA

TIME: ONE HOUR AND A HALF

1. Solve the simultaneous equations

$$x + y = a + b, \qquad \frac{y + b}{x + a} = \frac{a}{b},$$

and verify your results.

2. Solve the equation $x^2 - 1.6\,x - 0.23 = 0$, obtaining the values of the roots correct to three significant figures.

3. Write out the first four terms of $(a - b)^7$.

Find the fourth term of this expansion when

$$a = \sqrt[3]{x^{-1}y^{\frac{1}{2}}}, \qquad b = \sqrt[6]{9\,xy^{-4}},$$

expressing the result in terms of a single radical, and without fractional or negative exponents.

4. Reduce the following expression to a polynomial in a and b:

$$\frac{6\,a^3 + 7\,ab^2 + 12\,b^3}{3\,a^2 - 5\,ab - 4\,b^2} - \frac{1}{\dfrac{3}{19\,b} - \dfrac{5\,a + 4\,b}{19\,a^2}}.$$

5. The cost of publishing a book consists of two main items: first, the fixed expense of setting up the type; and, second, the running expenses of presswork, binding, etc., which may be assumed to be proportional to the number of copies. A certain book costs 35 cents a copy if 1000 copies are published at one time, but only 19 cents a copy if 5000 copies are published at one time. Find (*a*) the cost of setting up the type for the book, and (*b*) the cost of presswork, binding, etc., per thousand copies.

YALE UNIVERSITY

ALGEBRA A

TIME: ONE HOUR

Omit one question in Group II and one in Group III. Credit will be given for *six* questions only.

Group I

1. Resolve into prime factors: (*a*) $6x^2 - 7x - 20$; (*b*) $(x^2 - 5x)^2 - 2(x^2 - 5x) - 24$; (*c*) $a^4 + 4a^2 + 16$.

2. Simplify $\left(5 - \frac{a^2 - 19x^2}{a^2 - 4x^2}\right) \div \left(3 - \frac{a - 5x}{a - 2x}\right)$.

3. Solve $\frac{2(x-7)}{x^2 + 3x - 28} + \frac{2 - x}{4 - x} - \frac{x + 3}{x + 7} = 0$.

Group II

4. Simplify $\frac{\sqrt{2} + 2\sqrt{3}}{\sqrt{2} - \sqrt{12}}$, and compute the value of the fraction to two decimal places.

5. Solve the simultaneous equations $\begin{cases} x^{-\frac{1}{2}} + 2y^{-\frac{1}{2}} = \frac{7}{6}, \\ 2x^{-\frac{1}{2}} - y^{-\frac{1}{2}} = \frac{3}{6}. \end{cases}$

Group III

6. Two numbers are in the ratio of $c : d$. If a be added to the first and subtracted from the second, the results will be in the ratio 3 : 2. Find the numbers.

7. A dealer has two kinds of coffee, worth 30 and 40 cents per pound. How many pounds of each must be taken to make a mixture of 70 pounds, worth 36 cents per pound?

8. A, B, and C can do a piece of work in 30 hours. A can do half as much again as B, and B two thirds as much again as C. How long would each require to do the work alone?

CORNELL UNIVERSITY

ELEMENTARY ALGEBRA

1. Find the H. C. F.:

$$x^4 - y^4,$$
$$x^3 - xy^2 + x^2y - y^3,$$
$$x^4 + 2x^2y^2 - 3y^4.$$

2. Solve the following set of equations:

$$x + y = -1,$$
$$x + 3y + 2z = -4,$$
$$x - y + 4z = 5.$$

3. Expand and simplify:

$$\left(2x^3 - \frac{1}{x}\right)^7.$$

4. An automobile goes 80 miles and back in 9 hours. The rate of speed returning was 4 miles per hour faster than the rate going. Find the rate each way.

5. Simplify:

$$\frac{\left(\frac{x+1}{x-1}\right)^2 - 2 + \left(\frac{x-1}{x+1}\right)^2}{\left(\frac{x+1}{x-1}\right)^2 - \left(\frac{x-1}{x+1}\right)^2}.$$

6. Solve for x:

$$\frac{2x+3}{x-1} - 6 = \frac{5}{x^2 + 2x - 3}.$$

7. A, B, and C, all working together, can do a piece of work in $2\frac{2}{3}$ days. A works twice as fast as C, and A and C together could do the work in 4 days. How long would it take each one of the three to do the work alone?

UNIVERSITY OF CHICAGO

ELEMENTARY ALGEBRA

I. Resolve into simplest factors:

1. $a^2 - b^2 - (a-b)^2$.

2. $2a^3 - 3a^2 - 2a + 3$.

3. $x^4 - 14x^2y^2 + y^4$.

4. $a(a+b) - c(c+b)$.

II. Reduce to simplest form:

1. $-10\{x - 6[x - (y - z)]\} + 60\{y - (z + x)\}$.

2. $\dfrac{a}{(b-c)(b-a)} + \dfrac{b}{(c-a)(c-b)} + \dfrac{c}{(b-a)(c-a)}$.

3. $\left(\dfrac{a}{b^2} + \dfrac{b}{a^2}\right) \div \left(\dfrac{1}{a^2} - \dfrac{1}{ab} + \dfrac{1}{b^2}\right)$.

III. 1. Solve for x:

$\frac{1}{2}(x-5) - \frac{1}{6}(x-4) = \frac{1}{2}(x-3) - (x-2)$.

2. Solve for x and y:

$$\begin{cases} \frac{1}{8}(7+x) = \frac{1}{5}(9+y). \\ \frac{1}{7}(11+x+y) = \frac{1}{3}(9+y). \end{cases}$$

IV. Three brothers, A, B, C, at a family reunion were discussing their ages. C said to A: "Thirty years ago my age was double yours." Then B said to A: "Thirty-three years ago *my* age was double yours." If C's present age exceeds B's by three years and B's exceeds A's by seven years, find the age of each.

UNIVERSITY OF CALIFORNIA

ELEMENTARY ALGEBRA

1. If $a = 4$, $b = -3$, $c = 2$, and $d = -4$, find the value of:
   (a) $ab^3 - 3cd^2 + 2(3a - b)(c - 2d)$.
   (b) $2a^3 - 3b^4 + (4c^3 + d^3)(4c^2 + d^2)$.

2. Reduce to a mixed number:
$$\frac{3a^4 - 4a^3 - 10a^2 + 41a - 28}{a^2 - 3a + 4}.$$

Simplify:

3. $$\frac{a+2}{a^2+3a-40} - \frac{b-2}{ab-5b+3a-15}.$$

4. $$\left(1 - \frac{2-3b-2c}{a+2}\right) \div \frac{a^2-4c^2+9b^2+6ab}{2a^2+a-6}.$$

5. A's age 10 years hence will be 4 times what B's age was 11 years ago, and the amount that A's age exceeds B's age is one third of the sum of their ages 8 years ago. Find their present ages.

6. Draw the lines represented by the equations
$$3x - 2y = 13 \text{ and } 2x + 5y = -4,$$
and find by algebra the coördinates of the point where they intersect.

7. Solve the equations $\begin{cases} bx - ay = b^2 - ab, \\ y - b = 2(x - 2a). \end{cases}$

8. Solve $(2x+1)(3x-2) - (5x-7)(x-2) = 41$.

NEW YORK STATE EDUCATION DEPARTMENT

ELEMENTARY ALGEBRA

JANUARY, 1912

Answer the first six questions and two of the others.

1. Find the prime factors of *each* of the following: $1 - x^4$; $x^2 - cx - 2dx + 2cd$; $3a^5 + 3b^5$; $a^4 + a^2b^2 + b^4$.

2. Divide $4x^6 - 9x^4 + 25 - 14x^3 - x^2$ by $2x^3 - x - 5 + 3x^2$.

3. Solve $\begin{cases} mx - ny = m^2 + n^2, \\ x - y = 2n. \end{cases}$

4. Simplify $\sqrt{3} \times \sqrt{48}$; $(5\sqrt{5} - 4)(5\sqrt{5} + 8)$; $3\sqrt{8} \div 15\sqrt{2}$; $6\sqrt{\frac{3}{5}} - 5\sqrt{24} + 12\sqrt{\frac{3}{2}}$.

5. Find the square root of the following:

$$25x^4 - 30ax^3 + 49a^2x^2 - 24a^3x + 16a^4.$$

6. The area of a rectangle is 18 square inches less than twice the area of a square; the width of the rectangle equals the width of the square, but the length of the rectangle exceeds that of the square by 7 inches. Find the side of the square.

7. A, B, and C together have \$1285; A's share is \$25 more than $\frac{5}{6}$ of B's, and C's share is $\frac{4}{15}$ of B's. Find the share of each.

8. Solve $21x^2 = 2ax + 3a^2$.

9. Solve $\begin{cases} x^2 - xy + y^2 = 63, \\ x - y = -3. \end{cases}$

10. Define root of a number, surd, affected quadratic equation.

Expand $(2a^2 - 3b)^4$.

NEW YORK STATE EDUCATION DEPARTMENT

ELEMENTARY ALGEBRA

JUNE, 1912

Answer the first four questions and four of the others.

1. Divide $9x^6 - 10x^3 + 9 - 16x^2 - x^4$ by $4x - x^2 + 3x^3 - 3$. [Credit will not be granted if there is any error in the work.]

2. Factor *each* of the following: $x^4 - x^2 - 12$; $(a+2)^2 - 9x^2$; $x^3 - 27$; $6x^2 - x - 2$; $x^5 + 32$; $a^3 - a^2 - a + 1$.

3. Solve $\begin{cases} 2x + y = a + 1, \\ x - 2y = a - 1. \end{cases}$

4. The length of a rectangular field is twice its width; it costs as much to fence it at 50¢ per yard as it does to sod it at 15¢ a square yard. Find the dimensions.

5. Solve $2 + \sqrt{2x + 8} = 2\sqrt{x + 5}$.

6. Solve $x^2 + ax = 42a^2$.

Expand by the binomial formula $(2a - 3b)^5$, giving all the work.

7. Solve $\begin{cases} x^2 + y^2 = 13, \\ xy = 6. \end{cases}$

Group the four pairs of roots properly.

8. Two men start at the same time and travel in opposite directions; the ratio of their rates is 2 : 3 and in 5 hours they are 100 miles apart. Find the rate of each.

9. Divide the number c into *two* parts such that a times the larger part shall equal b times the smaller part.

10. If a triangle with equal sides has its sides increased 7 inches, 4 inches and 1 inch respectively, a right triangle is formed; find the sides of the right triangle.

NEW YORK STATE EDUCATION DEPARTMENT

ELEMENTARY ALGEBRA

JANUARY, 1913

Answer the first six questions and two of the others.

1. Factor each numerator and denominator in the following expressions; perform the operations indicated and reduce to simplest form:

$$\frac{x^4-9x^3-36x^2}{x^2a^2-10xa^2+9a^2}\times\frac{x^4-4x^2+3}{x^4-7x^2-18}\div\frac{x^3+x^2}{a^2x^2+2a^2}.$$

2. Solve $\begin{cases}\frac{x}{2}+\frac{y}{3}=4,\\ 5x-3y=2.\end{cases}$

3. Solve $x+\frac{a}{2}=\frac{a^2}{2x}$.

4. Simplify *each* of the following:

$$4\sqrt{24}+2\sqrt{54}-\sqrt{6}+3\sqrt{96}-5\sqrt{150}.$$

$$\sqrt[3]{375},\ \sqrt{\frac{2}{3}},\ \sqrt[3]{\frac{1}{4}},\ \frac{1}{\sqrt{3}}.$$

5. Solve $\frac{16x+3}{10}-\frac{2x-5}{5x-1}=\frac{8x-1}{5}$.

6. Separate 42 into *two* parts such that the greater part divided by the less shall give a quotient of 2 and a remainder of 3.

7. Solve $\begin{cases}6y^2-xy=2x^2,\\ 9y-12=-4x.\end{cases}$

8. What must be added to $x+a$ to make $y-b$?
What is the cost of 3 apples if a apples cost c cents?

9. The sum of two numbers is 8 and the sum of their cubes is 152; find the numbers.

10. If 1 is added to the numerator of a fraction the value of the fraction becomes $\frac{1}{2}$; if 1 is added to the denominator of the same fraction the value becomes $\frac{1}{3}$. Find the fraction.

NEW YORK STATE EDUCATION DEPARTMENT

ELEMENTARY ALGEBRA

JUNE, 1913

Answer the first six questions and two of the others.

1. Solve and check or prove $\frac{x}{3}-\frac{x-3}{3}=12-\frac{x+4}{2}-x$.

2. Extract the square root of $4c^4-4c^3+5c^2-2c+1$.

3. Solve $\begin{cases} x+2y=a, \\ 2x-y=b. \end{cases}$

4. Simplify $\sqrt{\frac{3}{4}}+\sqrt{\frac{1}{3}}$; $(\sqrt{5}-\sqrt{2})(2\sqrt{5}+3\sqrt{2})$.

5. In five years A will be twice as old as B; five years ago A was three times as old as B. Find the age of each at the present time.

6. Find the quotient to *three* terms and the remainder when $11a^3-5a+12-82a^2+30a^4$ is divided by $2a-4+3a^2$.

7. (*a*) What is the dividend which, divided by x, gives a quotient of y and a remainder of z?

(*b*) If a apples are sold for a dime, how many can be bought for c cents?

8. Two men, A and B, can dig a trench in 20 days; it would take A 9 days longer to dig it alone than it would B. How long would it take B alone?

9. Find *three* successive *even* numbers whose sum is $\frac{3}{8}$ of the product of the first two.

10. Simplify $\dfrac{x^2-\dfrac{1}{x}}{x+\dfrac{1}{x}+1}$.

THE UNIVERSITY OF THE STATE OF NEW YORK

ELEMENTARY ALGEBRA

JANUARY, 1910

Answer eight questions, selecting two from each group.

Group I

1. Solve $\frac{a+bx}{3b+2ax}=\frac{a-bx}{b-2ax}$.

2. Find the prime factors of *each* of the following expressions and from the factors determine the highest common factor:
$27m^5-8m^2$, $6m^3+8m^2-8m$, $12m^4-8m^3$, $27m^3-12m$.

3. Find the product of $2x+3y-z$ and $x-3y+2z$. Prove by substitution the correctness of the result if $x=1$, $y=2$, and $z=3$.

Group II

4. Reduce *each* of the following to its simplest form:
$\sqrt{50}-\sqrt{32}$, $2\sqrt{5}\times\sqrt{15}$, $6\sqrt{20}\div 2\sqrt{10}$, $\sqrt{\frac{5}{8}}$, $\sqrt{9\sqrt{a^4}}$.

5. Solve $\begin{cases} 2x+y-z=5, \\ 3x+2y-4z=2, \\ x-2y+3z=3. \end{cases}$

6. Find the square root of the following:
$49x^6-42x^5-47x^4-4x^3+28x^2+16x+4$.

Group III

7. If 122 marbles were divided among three boys so that the first had twice as many as the second and the second had 6 more than the third, how many had each? Prove.

8. Find *three* consecutive numbers such that if the first is divided by 5, the second by 7, and the largest by 11, the sum of the three quotients is $\frac{1}{7}$ of the sum of the three numbers. Prove.

9. Solve $\sqrt{2x+1} = 2\sqrt{x} - \sqrt{x-3}$.

Group IV

10. Solve $\begin{cases} 2x + y = 5, \\ 3x^2 - 7y^2 = 5. \end{cases}$

11. Solve $1 - 10ax + 16a^2x^2 = 0$.

12. A and B can together address 100 envelopes in an hour; when each works alone A can address 100 envelopes in 50 minutes less time than B. How many can each address in an hour?

THE UNIVERSITY OF THE STATE OF NEW YORK

ELEMENTARY ALGEBRA

JANUARY, 1915

Answer 12 questions, selecting five from group I, two from group II, and five from group III.

Group I

1. Solve $\dfrac{3(x-2)}{x+3} = \dfrac{36-4x}{x^2-9} - \dfrac{2+3x}{3-x}$.

2. Factor *three* of the following:

$$x^4 - 8x^2 - 9,$$
$$m^2 - 6mn - 16x^2y^2 + 9n^2,$$
$$m^2d^2 + 3 - 3m - d^2,$$
$$p^2q^2 - 12pqx + 35x^2.$$

3. Solve $\sqrt{3+x} + \sqrt{x} = \dfrac{5}{\sqrt{x}}$.

4. Solve $3(x-2)(x-4) = (x-5)^2$.

5. (*a*) Simplify $2\sqrt{\frac{5}{3}} - \sqrt{60} - 5\sqrt{\frac{3}{5}}$.

   (*b*) Simplify $\dfrac{a\sqrt{x} - 2b\sqrt{y}}{a\sqrt{x} + b\sqrt{y}}$.

6. Solve $\begin{cases} x - 3y = 1, \\ xy + y^2 = 5. \end{cases}$

Group II

7. A man has \$8000 which he wishes to invest in two enterprises so that his total income will be \$425; if one enterprise pays $5\frac{1}{2}$ % and the other 5 %, how much must he invest in each?

8. Separate a line 20 inches long into two parts such that the product of the whole line and one part shall equal the square of the other part. [Result contains a surd.]

9. A rectangular ceiling has in it two skylights each $2\frac{1}{2}$ feet by 3 feet; the surface of the ceiling, not including the skylights, is 93 feet. If the length of the ceiling is 3 feet more than its width, what are its dimensions?

Group III

10. If one of the factors of $6a^2x^2 - 4a^3x - 4ax^3 + x^4 + a^4$ is $a^2 + x^2 - 2ax$, what is the other factor?

11. Write *three* different expressions of higher degree than the first degree whose H. C. F. is $x - y$. Find the L. C. M. of these expressions.

12. A lady bought 5 dozen buttons at d cents a dozen and 3 yards of cloth at k cents a yard; she gave a two-dollar bill in payment. How many cents should she receive in change?

13. What is the value of $8x^2 - 6ax$ when $x = \frac{a - \sqrt{b}}{2}$?

14. Find, correct to *two* decimal places, the solutions of $2x^2 + 6x - 3 = 0$.

15. Write an expression that can be divided by $a - b$ and also by $2a + b$.

THE UNIVERSITY OF THE STATE OF NEW YORK

ELEMENTARY ALGEBRA

JUNE, 1915

Answer the first eight questions and two of the others.

1. Find the prime factors of *each* of the following:

$$27 - 64x^3;\ 10x^2 - 7x - 6;\ ax^2 - cx + ax - c.$$

2. When $a=2$, $b=3$, $c=-4$, find the value of the following:

$$(3a^2 - c^2)(a + c)\sqrt{(7a + c)(4b - a)}.$$

3. Solve $x^2 - 4x - 1 = 0$; find the roots correct to *two* places of decimals.

4. Solve $\sqrt{x + 16} - \sqrt{x} = 2$.

5. (*a*) Simplify $\sqrt{48} - 2\sqrt{45} + 10\sqrt{\frac{1}{5}} - \sqrt{\frac{1}{3}}$.

(*b*) Simplify $(3\sqrt{5} - 2\sqrt{2})(2\sqrt{5} + 4\sqrt{2})$.

6. Solve $\begin{cases} ax = b(y - 2), \\ y - x = \dfrac{a^2 + b^2}{ab}. \end{cases}$

7. Solve $\begin{cases} 2x + y = 1, \\ 3x^2 + 12x + 4y = 8. \end{cases}$

8. A man has oats enough for x horses for y days; how long will the oats last m horses?

9. If the greater of two numbers is divided by the less, the quotient is 2 and the remainder is 1. If the less is increased by 20 and this result is divided by the greater increased by 3, the quotient is 2. Find the numbers.

10. What must be the value of m in order that

$$2x^2 - 3x^3 + 21x + x^4 + 3m$$

may be exactly divisible by $x^2 - 3 + x$?

11. (*a*) What value of x will make $(6x - 5)(2x - 3)$ equal to 13 more than $(3x + 2)(4x - 1)$?

(*b*) How many dimes taken from a dollars will leave $10x$ cents?

THE UNIVERSITY OF THE STATE OF NEW YORK

ELEMENTARY ALGEBRA

JANUARY, 1914

Answer the first six questions and two of the others.

1. Find the prime factors of $x^2 - x - 30$; $16 - 2m^3$; $1 - x^4$; $2ab - cx + 2c - abx$.

2. Subtract $-5x - 2(6y - 2z)$ from $8x - \overline{z + 2y}$ and add the result to $6z - (3y - 4x)$.

3. Simplify $\left(\frac{2}{x+y}\right)\left(\frac{2x^2}{x^2-y^2} + \frac{x+y}{2(x-y)} - \frac{x-y}{2(x+y)}\right)$.

4. Solve for a and b $\begin{cases} a - \frac{3b}{2} = \frac{1}{2}. \\ 3a + 7b = 13. \end{cases}$

5. Solve $(x-6)^2 - (2x-5)^2 = 16$.

6. One number is twice another number; when the smaller is subtracted from 32 the remainder is 11 less than the remainder when the larger is subtracted from 50. Find the numbers.

7. Solve and test

$$3(x+1)(x-3) - (3-x)^2 = 8(2x-3) + 2x^2 - 26.$$

8. (*a*) Reduce to radicals of the same degree $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[6]{2}$.

(*b*) Perform the indicated operations $(2 - \sqrt{5})^2(1 - 2\sqrt{5})$; $2\sqrt{6} \div \sqrt[3]{3}$.

9. The length of a rectangle is 15 ft. greater than its width; if each dimension is decreased 2 ft., the area will be decreased 106 sq. ft. Find the dimensions.

10. (*a*) The dividend is m, the divisor is n, the remainder is r; what is the quotient?

(*b*) A house cost a dollars and rents for n dollars a month; what per cent per annum is the income of the investment?

THE UNIVERSITY OF CHICAGO

ELEMENTARY ALGEBRA

I. Remove the symbols of aggregation from:

1. $2x-\{3y-[5z-(x-\overline{2y-3z})]\}$;
2. $2x+\{5z-[3y+(x-\overline{2y-3z})]\}$.

II. Arrange the following in groups of equivalent expressions:

$$\frac{x}{y},\ \frac{x+a}{y+a},\ \frac{xa}{ya},\ \frac{x-a}{y-a},\ \frac{x\div a}{y\div a};$$

$$\frac{\frac{x}{a}+1}{\frac{y}{a}+1},\ \frac{1-\frac{a}{x}}{\frac{y}{x}-\frac{a}{x}},\ \frac{\frac{x}{y}+\frac{a}{y}}{1+\frac{a}{y}},\ \frac{\frac{x}{a}-1}{\frac{y}{a}-1}.$$

III. Find the G. C. D. of:

$2ax^3+2ax^2-4ax$,
$(ax^2+2ax)(2x^3-2)$,
$2a(x^2-x)(x^3+8)$,
$(x-1)(6abx^2+12abx-2ax^2y-4axy)$.

IV. Solve for x:

$$\frac{2x-1}{2x-2}-\frac{23}{10x-10}=\frac{3}{5}\left(\frac{1}{x-1}-\frac{1}{3}\right).$$

V. Two yachts race over a 48-mile course. Owing to difference in measurement, B is given a start of half a mile in the first trial, and is beaten by 6 minutes. In the second trial, the rate of the wind being the same as before, B's start is increased to a mile and a half, and still A wins by 2 minutes. Find the rate in feet per minute of each boat.

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$[(x+y)-(m+n)]^2 =$

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C+	a+
a+	a+
a+	a+
B	a+
a+	a+
a+	a+
a+	a+
a+	a+
a+	a+
B-	C-
a+	a+
a (test)	a+